Mathematical Reviews

Vol. 4, No. 8 September, 1943 pp. 209–236

TABLE OF CONTENTS

THE MICROFILM SERVICE

Through the services of the Brown University Photographic Laboratory one can obtain, either on microfilm or as photoprint, the complete text of any article listed in MATHEMATICAL REVIEWS or available in the Brown University Library, except books and material of which the reproduction is prohibited by copyright laws.

Procedure for Ordering Microfilm or Photoprint

1. Give your name and address.

2. State whether you wish microfilm or photoprint.

3. (a) If the article in question has been listed in MATHEMATICAL REVIEWS, give the MF number printed at the end of the title. Titles without MF numbers are copyrighted.

   (b) If the article has not been reviewed (for instance because published prior to 1939) give as accurately as possible the author, title, journal, volume, year and pages.

4. Enclose cash for the amount of the order. U. S. currency or check, made payable to Brown University, will be accepted. The minimum charge is 45¢.

5. Mail to: *Brown University Photographic Laboratory, Providence, Rhode Island.*

Cost

MICROFILMS are furnished for

15¢ for articles with less than 25 pages.

30¢ for articles with more than 24 pages, but less than 50 pages.

50¢ for articles with more than 49 pages, but less than 100 pages.

\$1.00 for articles with more than 99 pages.

PHOTOPRINTS cost 16¢ per sheet, each sheet containing two pages of text, except in the case of unusually large journals (larger than 7" by 10"). In the latter case you will be informed of the extra charge before the order is filled, unless you have indicated on your order that the reprint is desired even at the higher price of 16¢ per page.

MINIMUM CHARGE. If an order totals less than 45¢ the cash included should be raised to 45¢ in order to cover the basic expenses of this service.

Microfilm and photoprints with MF numbers show black writing on white background. Articles which are not reviewed, and which therefore do not have an MF number, are reproduced only in negative films or prints, i.e. white letters on black background. Microfilms can be read in any reading machine adapted to standard 35 mm. double perforated film. Photoprints are on sheets 10" by 14" (about 25 by 35 cm.).

ONE-SIDE EDITION OF MATHEMATICAL REVIEWS

An edition of MATHEMATICAL REVIEWS printed on only one side of the paper is available to persons interested in making card files of reviews or wishing to add remarks to reviews in the future. This special edition may be obtained for an additional payment of \$1.00. A regular current subscription can be changed to a one-side subscription by paying the additional \$1.00. This edition is folded but not stitched.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
ACADEMIA NACIONAL DE CIENCIAS EXACTAS, FISICAS Y NATURALES DE LIMA
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY
UNION MATEMATICA ARGENTINA

Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

Subscriptions: Price \$13 per year (\$6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.

Mathematical Reviews

Vol. 4, No. 8 SEPTEMBER, 1943 Pages 209-236

FOUNDATIONS

★**Carnap, Rudolf. Introduction to Semantics.** Harvard University Press, Cambridge, Mass., 1942. xii+263 pp. $3.50.

This is the first volume of a series entitled "Studies in Semantics." It describes the fundamental ideas of the author's theory of semantics and forms the basis for all later volumes of the series. The author believes that the semantics of formal systems, that is, their meaning and interpretation, should be studied by the same precise methods as the syntax of such systems. This parallelism between semantics and syntax is emphasized by the arrangement of the book. The first three chapters deal with semantical concepts, the fourth with the corresponding syntactical concepts and the last chapter with relations between semantics and syntax, and in particular with the notion of a true interpretation of a formal system or calculus. Truth is taken to be the fundamental semantical concept. It may be supposed that the meaning of a sentence is determined by a knowledge of all conditions under which the sentence is true. The author finds it necessary to distinguish between logical truth and factual truth, and this distinction is the source of much complication of detail. Of several possible theories of logical truth, one theory is developed at length which depends on the notions of "L-states," and the "L-range" of a sentence. For example, a sentence L-implies (logically implies) another if the L-range of the first is included in the L-range of the second. Here the L-range of a sentence means the set of all L-states admitted by the sentence. This notion of logical truth is suggestive of the modal concepts such as necessarily true and possibly false of C. I. Lewis' systems of strict implication. (Such modal logics are to be the subject of the third volume of the series.)

Many concepts in this book occur in quadruplicate. For example, there is "true" (this is called a radical concept), "L-true" (that is, logically true), "F-true" (factually true), and "C-true." The latter means provable, or derivable in a calculus, and is a syntactical concept, rather than a semantical one. An important notion in this theory is that of a semantical system S. This means an object language K (such as a propositional calculus), together with a set of semantical rules stated in the metalanguage which determine the truth conditions of each sentence of K. The semantical system S is said to be a true interpretation of K if, whenever a formal relationship holds in K between sentences, the corresponding semantical relationship holds in S between the same sentences. In particular, if a sentence is provable in K, it must be true in S. (Note that the converse is not assumed; a sentence may be true in S without being provable in K.) The use of truth-tables as semantical rules is also explained.

Since the terminology of this book is to be used in later volumes, care is taken to define all technical terms as exactly as possible, and to illustrate them by examples. There is an appendix in which questions of terminology are discussed. The appendix also contains an outline of further semantical problems, and a translation of certain terms used by the author in previous works into the terminology of this book. *O. Frink* (State College, Pa.).

★**Carnap, Rudolf. Formalization of Logic.** Harvard University Press, Cambridge, Mass., 1943. xviii+159 pp. $3.00. 1943

This is the second volume of a series; it presupposes a familiarity with the first volume [see the preceding review]. The author applies his semantical method to the propositional calculus and to the functional calculus of first order. He concludes that these calculi do not constitute a full formalization of the corresponding logical theories, but that this defect can be remedied by enlarging the calculi to include the notion of junctive. A junctive is a class of sentences taken in one of two ways, conjunctively or disjunctively.

The ordinary propositional calculus PC fails to be a full formalization of the logic of propositions because it has nonnormal true interpretations. A semantical system S is said to be a normal interpretation of a propositional calculus if the semantical rules for the logical connectives in S are in agreement with the usual truth-tables for these connectives. It is shown by examples that PC has two distinct types of nonnormal true interpretation. In the second type, for instance, two sentences of S may both be false, while their disjunction is true. But when PC is extended to include the notion of junctive, with the proper syntactical and semantical rules, these nonnormal interpretations are excluded. For example, if two sentences are both false in an interpretation S, so is their disjunctive by one of the semantical rules for junctives. By one of the syntactical rules, their disjunction implies their disjunctive. Hence the former cannot be true if S is a true interpretation, and the second type of nonnormal interpretation is ruled out.

The last chapter describes a similar modification of the functional calculus of first order to include the notion of transfinite junctive. After defining what is meant by normal interpretations of the quantification operators, the author shows that the extended functional calculus with junctives has no nonnormal interpretations and is consequently a full formalization of the logic of functions. An equivalent method of extending the functional calculus which avoids the use of junctives is also outlined. This is based on the concept of the relation of involution between sentential classes.

This book is of interest both as a contribution to formal logic and as the first detailed illustration of the author's theory of semantics. *O. Frink* (State College, Pa.).

Post, Emil L. Formal reductions of the general combinatorial decision problem. Amer. J. Math. 65, 197–215 (1943). [MF 8197]

This paper is devoted to a proof that every system of mathematical logic in a certain canonical form may be formally reduced to a system in normal form. Roughly, a system is in canonical form if (i) its primitive assertions

constitute a specified finite set of enunciations (that is, formulae of the system), each of which is a finite sequence of letters, the different letters constituting a given finite set, (ii) the operations of the system are a specified finite set of productions, each of the form:

$$g_{11}P'_{i_1}g_{12}P'_{i_2}\cdots g_{1m_1}P'_{i_{m_1}}g_{1(m_1+1)},$$
$$g_{21}P''_{i_1}g_{22}P''_{i_2}\cdots g_{2m_2}P''_{i_{m_2}}g_{2(m_2+1)}, \quad \cdots,$$
$$g_{k1}P^{(k)}_{i_1}g_{k2}P^{(k)}_{i_2}\cdots g_{km_k}P^{(k)}_{i_{m_k}}g_{k(m_k+1)},$$

produce $g_1P_{i_1}g_2P_{i_2}\cdots g_mP_{i_m}g_{m+1}$, where the g's represent specified sequences of the primitive letters of the system (including the null sequence), the P's represent the operational variables of the production, each premise and each conclusion has at least one operational variable, the conclusion is not null and every operational variable in the conclusion is present in at least one premise. A system in canonical form, then, is said to be in normal form if it has but one primitive assertion, and each of its productions has the form: gP produces Pg'. The assertions of the system are the primitive assertions together with all the enunciations obtained from the primitive assertions by repeated applications of the productions.

The reduction of a system in canonical to one in normal form involves four successive reductions: (1) the reduction of the system to one in which there is but one primitive assertion, and in which each production involves a single premise; (2) the reduction of the system thus obtained to one in which all the productions have a certain special form; (3) the reduction of the system obtained by (2) to one in which all of the productions involve but a single operational variable; and (4) the reduction of the system thus obtained to one in normal form.

As a result of this theorem that canonical may be reduced to normal systems, it follows that any procedure, which could effectively determine for an arbitrary enunciation of a system in normal form whether or not it is an assertion of that system, could be reproduced for the system in canonical form. It can be shown, Post remarks, that the problem of deciding whether a well-formed formula of the Church calculus of lambda-conversion has or has not a normal form can be reduced to the decision problem of a system in canonical form. It follows then that no effective procedure exists for determining whether any arbitrary enunciation of any arbitrary system in normal form is an assertion of the system or not.

The notions employed in this paper afford an independent approach to the problem of unsolvability, and the author conjectures that by their means further light may be cast upon the unsolvability of hitherto unsolved decision problems. *R. M. Martin* (Cambridge, Mass.).

Greenwood, Thomas. La valeur des géométries non-Euclidiennes. Rev. Trimest. Canad. **29**, 113–131 (1943). [MF 8356]

The author sketches briefly the historical development of the hyperbolic and elliptic geometries, from attempts to prove the parallel postulate to the approaches furnished by differential and projective geometry, and emphasizes that the non-Euclidean geometries are, as abstract systems, on quite the same footing as the more familiar geometry of Euclid. *L. M. Blumenthal* (Columbia, Mo.).

NUMBER THEORY

Heath, R. V. A magic cube with $6n^3$ cells. Amer. Math. Monthly **50**, 288–291 (1943). [MF 8330]

An ordinary magic cube of order n ($n>3$) is defined as an arrangement of n^3 different integers in a three-dimensional matrix so that the sum of the numbers in every row, every column, every file and each diagonal of every horizontal layer, as well as in the four space-diagonals, shall be the same number S. To make a model, we may use n^3 small solid cubes, each marked with the appropriate number on one of its faces, say the top face. However, since each of the small cubes has six faces, it is natural to ask whether we can mark all the $6n^3$ faces with different numbers (namely, the integers from 1 through $6n^3$) so that, when the whole figure is rotated six ways, each face in turn becoming the top face, we have in every case a magic cube (with $S=(6n^3+1)n/2$). This is clearly impossible for odd values of n. An example where $n=4$ is given in detail, with the extra property that the pairs of numbers on opposite faces of all the small cubes have the constant sum $6n^3+1=385$. *H. S. M. Coxeter.*

Pillai, S. S. On the divisors of a^n+1. J. Indian Math. Soc. (N.S.) **6**, 120–121 (1942). [MF 8293]

Using two simple lemmas concerning congruences and quadratic residues, the author proves that no prime of the form $4am-1$ divides a^n+1 for any n. *R. D. James.*

Pillai, S. S. On a congruence property of the divisor function. J. Indian Math. Soc. (N.S.) **6**, 118–119 (1942). [MF 8292]

Let $d(n)$ denote the number of divisors of n and $N(k, x)$ the number of $n\leqq x$ for which $d(n)$ is a multiple of k. The author gives a short proof of the results

$$N(k, x)=Ax+O(x^{1/(k-1)}\log x), \quad k\geqq 3, \text{ prime},$$
$$N(2, x)=[x]-[\sqrt{x}].$$

Here

$$A=\sum_{q,r}((-1)^{f(q)-1}\varphi(q)/q^{rk}),$$

where $f(q)$ is the number of different prime factors of q and $\varphi(q)$ is the Euler φ-function. *R. D. James.*

Zuckerman, Herbert S. On some formulas involving the divisor function. Bull. Amer. Math. Soc. **49**, 292–298 (1943). [MF 8227]

This paper begins with elementary number theory proofs of two formulas given by Viggo Brun. They are

(1) $T_1(n)-T_2(n)+T_3(n)-\cdots=\mu(n)$,

(2) $h(n)=T_1(n)-\frac{1}{2}T_2(n)+\frac{1}{3}T_3(n)-\cdots=\begin{cases}1/t, & n=p^t,\\ 0, & \text{otherwise},\end{cases}$

where $T_l(n)$ is the number of ways that n can be expressed as a product of l factors each greater than 1. The proof of (1), for example, is made to depend on the relation

$$T_{l+1}(n)=\sum_{d|n}T_l(n)-T_l(n)$$

and the Mobius inversion formula. The same method of proof leads to extension of (1), such as

$$\sum_{l=0}^{\infty}(-1)^l(l+1)T_l(n)=\sum_{d|n}\mu(d)\mu(n/d)$$
$$=\begin{cases}1 & \text{if } n=1,\\ (-1)^s2^s & \text{if } n=p_1\cdots p_sq_1^2\cdots q_t^2,\\ 0 & \text{if } n \text{ is divisible by a cube.}\end{cases}$$

Other similar formulas are obtained by combining (1) and (2). *R. D. James* (Saskatoon, Sask.).

Gupta, Hansraj. A formula in partitions. J. Indian Math. Soc. (N.S.) **6**, 115–117 (1942). [MF 8291]

In this paper $P_m(n)$ denotes the number of partitions of n in which the smallest summand is m and $p_m(n)$ denotes the number of partitions in which the largest summand is m. The relation

$$P_{m+1}(n+2m+1)=p_1(n)+p_2(n-m)+p_3(n-2m)+\cdots+p_{r+1}(n-rm), \quad r=[(n-1)/(m+1)],$$

together with the known inequalities for $p_m(n)$ and similar ones for $P_m(n)$, lead to the asymptotic formula

$$P_m(jm+m+s)\sim\frac{1}{j!}\binom{u}{j-1}+\frac{1}{(j-1)!}\binom{m+u+c_j}{j-2}$$

for $j=o(\log m)$, where $u=s+j(j+1)/2$, $c_j=(j-1)2^{j-3}$.
R. D. James (Saskatoon, Sask.).

Auluck, F. C., Chowla, S. and Gupta, H. On the maximum value of the number of partitions of n into k parts. J. Indian Math. Soc. (N.S.) **6**, 105–112 (1942). [MF 8289]

Denote by $P_k(n)$ the number of partitions of n into exactly k parts and by $p_k(n)$ the number of partitions of n which have at most k summands. The latter function has been studied by Erdös and Lehner [Duke Math. J. **8**, 335–345 (1941); these Rev. **3**, 69], who derived an asymptotic formula for $p_k(n)/p(n)$, where $p(n)$ is the number of unrestricted partitions of n. Theorem 1 in the present paper is a similar result for $P_k(n)$, namely,

$$\lim_{n\to\infty} n^{\frac12}P_k(n)/p(n)=\exp\{-\tfrac12 Cx-(2/C)e^{-\frac12 Cx}\},$$

where $k=C^{-1}n^{\frac12}\log n+xn^{\frac12}$, $C=\pi(\frac23)^{\frac12}$. A proof is given following the method of Erdös and Lehner but without using their explicit results. Theorem 2 concerns a maximum for $P_k(n)$ regarding n as fixed and k variable. The authors present evidence that there exists an integer k_0 such that

$$P_k(n)\geqq P_{k-1}(n), \qquad k\leqq k_0,$$
$$P_k(n)\leqq P_{k-1}(n), \qquad k\geqq k_0.$$

They are, however, able to prove only the following result. If $P_k(n)\leqq P_{k_1}(n)$ if $k\neq k_1$ then, for $n>n_0$,

$$n^{\frac12}<k_1<\delta n^{\frac12}\log n,$$

where δ is any fixed number greater than $1/C$. The proof is in two parts and consists of showing by different methods that $n^{\frac12}P_k(n)/p(n)=o(1)$ for $k>B^{-1}n^{\frac12}\log n$ and for $k\leqq n^{\frac12}$, where $B<C$. *R. D. James* (Saskatoon, Sask.).

Auluck, F. C. An asymptotic formula for $p_k(n)$. J. Indian Math. Soc. (N.S.) **6**, 113–114 (1942). [MF 8290]

An elementary proof of the result due to Erdös and Lehner, and also proved by H. Gupta [Proc. Indian Acad. Sci., Sect. A. **16**, 101–102 (1942); these Rev. **4**, 190], that

$$p_k(n)\sim\tbinom{n-1}{k-1}/k!$$

R. D. James (Saskatoon, Sask.).

Sispánov, S. The sieve of Eratosthenes and the logarithmic integral of Tchebysheff. Bol. Mat. **15**, 105–116 (1942). (Spanish) [MF 7540]

After eliminating multiples of the first k primes, the sequence of the remaining positive integers 1, $p_1, \cdots, p_k, p_{k+1}, \cdots, p_n, p_{k+1}^2, \cdots$ is said to have (approximate) density $\delta_k=\prod p_i/(p_i-1)$ $(i=1, \cdots, k)$. Consider now a graph having as ordinates the primes p_n for abscissae $n=1, 2, \cdots$. This is approximated by a continuous differentiable curve with slopes at the points with $p=p_k^2$ determined by $dp/dn=\delta_k$. From this is obtained $dp/dn=\ln p$, whence $n=\int_2^p dx/\ln x$. The closeness of the approximation is not investigated. *G. Pall* (Montreal, Que.).

Vinogradow, I. M. On the estimation of trigonometrical sums. C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 182–183 (1942). [MF 7600]

Several improvements are stated of results in Vinogradow's book [A New Method in the Analytical Theory of Numbers, Trav. Inst. Math. Stekloff, vol. 10, 1937]: (i) on an upper bound to $\sum\exp(2\pi iF(x))$ $(x=Q+1, \cdots, Q+P)$, where $f(x)=a_nx^n+\cdots+a_1x$, $a_n=a/q+\theta/q^2$, etc.; (ii) and (iii) Hardy and Littlewood's asymptotic formula for the number of representations by $x_1^n+\cdots+x_r^n$ $(n>10)$ and Vinogradow's similar formula for $p_1^n+\cdots+p_r^n$ both hold for $r\geqq 20n^2\log n$; (iv) the remainder in the formula for $\pi(N)$ does not exceed $N\cdot\exp(-(\log N)^{0.6+\epsilon})$. Finally, lemma 6 of Vinogradow [Rec. Math. [Mat. Sbornik] N.S. **3(45)**, 435–470 (1938)] can be improved, it is stated, to yield $T<40p^{r-\frac12 n(n+1)}$. *G. Pall* (Montreal, Que.).

Vinogradow, I. An improvement of the estimation of trigonometrical sums. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **6**, 33–40 (1942). (Russian. English summary) [MF 7695]

The proof of a result similar to that in (i) of the preceding review is given. As an application an improved estimate is obtained of the number of integers x between 1 and P for which the fractional part of $f(x)$ does not exceed a given number r. *G. Pall* (Montreal, Que.).

Linnik, U. V. On Weyl's sums. C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 184–186 (1942). [MF 7601]

A proof is outlined that $|S|\leqq P^{1-s}$, $s=1/(22400n^2\ln n)$, where $S=\sum_{x=1}^{P}\exp 2\pi i\alpha f(x)$, $\alpha=a/q+\theta/q^2$, $(a, q)=1$, $|\theta|\leqq 1$, $f(x)=a_0x^n+\cdots+a_n$, $P\leqq q\leqq P^{n-1}$. The proof is based on an upper bound $p^{s-\frac12 n(n+1)+1/n^{10}}$ to the number of solutions x_j of the system of Diophantine equations $x_1^i+\cdots+x_s^i=M_i$ $(i=1, \cdots, n)$, where the x_j assume certain sets of prime values. *G. Pall* (Montreal, Que.).

Linnik, U. New estimations of Weyl's sums by the method due to Vinogradow. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **6**, 41–70 (1942). (Russian. English summary) [MF 7696]

Results are proved similar to that stated in the preceding review, but having $n^{14/3}$ in place of n^2. *G. Pall.*

Linnik, U. V. On the representation of large numbers as sums of seven cubes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 162 (1942). [MF 7615]

A proof that $G(3)\leqq 7$ is announced. The proof is stated to depend on Linnik's result on large numbers represented by certain ternary quadratic forms [Bull. Acad. Sci. URSS. Sér. Math. **4**, 363–402 (1940); these Rev. **2**, 348], an estimate by B. I. Segal of the number of integers containing only two given primes as factors [C. R. (Doklady) Acad. Sci. URSS (N.S.) **1933**, 47–49 (1933)] and theorems on the distribution of primes in arithmetical progressions.
G. Pall (Montreal, Que.).

Davenport, H. On the product of three homogeneous linear forms. IV. Proc. Cambridge Philos. Soc. 39, 1–21 (1943). [MF 7964]

Part I of the series appeared in J. London Math. Soc. 13, 139–145 (1938); part II in Proc. London Math. Soc. (2) 44, 412–431 (1938); part III in the same Proc. 45, 98–125 (1941). A paper in J. London Math. Soc. 16, 98–101 (1941) [these Rev. 3, 70] is also connected with the present one. These papers will be referred to as [1], $\cdots$, [4], respectively.

Let L_1, L_2, L_3 be three real linear forms in u, v, w of determinant 1, and let M denote the lower bound of $|L_1L_2L_3|$ for integral u, v, w not all zero. The author previously [2] proved that $M\leqq 1/7$. In the present paper he develops the method of a later proof of his [4] and so shows: "Either $M=1/7$ and L_1, L_2, L_3 are equivalent in some order to
$$\lambda_1(u+\theta v+\phi w),\quad \lambda_2(u+\phi v+\psi w),\quad \lambda_3(u+\psi v+\theta w),$$
where θ, ϕ, ψ are the roots of $t^3+t^2-2t-1=0$ and $|\lambda_1\lambda_2\lambda_3|=1/7$, or $M=1/9$ and L_1, L_2, L_3 are equivalent in some order to
$$\lambda_1(u+\theta'v+\phi'w),\quad \lambda_2(u+\phi'v+\psi'w),\quad \lambda_3(u+\psi'v+\theta'w),$$
where θ', ϕ', ψ' are the roots of $t^3-3t-1=0$ and $|\lambda_1\lambda_2\lambda_3|=1/9$, or $M<1/9.1$."

For the proof, it can be assumed that $M\geqq 1/9.1$ and that
$$L_i=L_i^*(u+\alpha_i v+\beta_i w),$$
where
$$|L_1^*L_2^*L_3^*|=\frac{M}{1-\epsilon},\quad \begin{vmatrix}1&\alpha_1&\beta_1\\1&\alpha_2&\beta_2\\1&\alpha_3&\beta_3\end{vmatrix}=\mp\frac{1-\epsilon}{M},$$
and where $\epsilon>0$ is arbitrarily small; then α_i, β_i satisfy the inequalities
$$\left|\prod_{i=1}^{3}(u+\alpha_i v+\beta_i w)\right|\geqq 1-\epsilon$$
for integral u, v, w not all zero. It may further be assumed that the positive definite quadratic form
$$\{(\alpha_1-\alpha_2)v+(\beta_1-\beta_2)w\}^2+\{(\alpha_2-\alpha_3)v+(\beta_2-\beta_3)w\}^2+\{(\alpha_3-\alpha_1)v+(\beta_3-\beta_1)w\}^2=2(Av^2+Bvw+Cw^2)$$
is reduced: $|B|\leqq A\leqq C$; for sufficiently small ϵ, its determinant satisfies
$$4(AC-B^2)=3((1-\epsilon)/M)^2<248.5.$$
These conditions restrict the triples of numbers (α_i), (β_i) to certain types, according to the integral parts of α_i, β_i, $\alpha_i+\beta_i$, $\alpha_i-\beta_i$; all but two of these cases can be excluded.

K. Mahler (Manchester).

Artin, Emil and Scherk, Peter. On the sum of two sets of integers. Ann. of Math. (2) 44, 138–142 (1943). [MF 8280]

Let A, B be sets of nonnegative integers a, b, respectively. Let $C=A+B$ be the set of all integers of the form $a+b$. Let $A(x)$, $B(x)$, $C(x)$ denote the number of positive integers of the sets not greater than x. The authors prove the following. Theorem I. Let $n \not\subset C$. Then
$$C(n)-C(n-m)=A(m-1)+B(m-1)+Z_m$$
for a suitable $m\not\subset C$, $0<m\leqq n$, where Z_m denotes the number of decompositions of m of a certain type. From this result follows theorem II: if $C(n)<n$, then
$$C(n)-C(n-m)\geqq A(m-1)+B(m-1)$$
for a suitable $m\not\subset C$ with $0<m\leqq n$. Two consequences of this result are a theorem of H. B. Mann [Ann. of Math. (2) 43, 523–527 (1942); these Rev. 4, 35] and one of A. S. Besicovitch [J. London Math. Soc. 10, 246–248 (1935)]. The methods of proof which the authors use are simplifications of those of Mann. *B. W. Jones* (Ithaca, N. Y.).

Mahler, K. Note on lattice points in star domains. J. London Math. Soc. 17, 130–133 (1942). [MF 8257]

This note is a summary of results of which proofs will appear in the Proceedings of the London Mathematical Society. The author, generalizing Minkowski's original methods, obtains theorems on lattice points in nonconvex domains. He treats the general star region K, a closed bounded point set satisfying (a) K contains the origin O of the coordinate system (x, y) in its interior; (b) the boundary L of K is a Jordan curve consisting of a finite number of analytic arcs; (c) every radius vector from O intersects L in one and only one point. A lattice Λ of points P
$$(x, y)=(\alpha h+\beta k, \gamma h+\delta k),\quad h, k=0, \pm1, \pm2, \cdots,$$
is said to be K-admissible if the origin O is the only point of Λ interior to K. Let $d(\Lambda)=|\alpha\delta-\beta\gamma|$, and let $\Delta(K)$ be the lower limit of $d(\Lambda)$ for all K-admissible lattices. Then $\Delta(K)>0$, and the author states that there exists at least one K-admissible lattice Λ such that $d(\Lambda)=\Delta(k)$ (a critical lattice in Mordell's terminology). Furthermore, he has found an algorithm by means of which all critical lattices of K can be determined in a finite number of steps, and he lists a few special cases in which the algorithm has been applied in detail. *D. C. Spencer.*

ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

Cuesta Dutari, Norberto. Generalized real numbers. Revista Mat. Hisp.-Amer. (4) 2, 5–12, 62–66, 104–109, 218–225 (1942). (Spanish) [MF 7080]

Generalized real numbers are transfinite sequences of the integers 0, 1, $\cdots$, 9 written $0.a_1a_2\cdots a_\beta\cdots$, $\beta<\omega\cdot\epsilon$. The sequences are ordered lexicographically, hence well-ordered, and one identifies the sequences $0.a_1a_2\cdots a_{\beta-1}9_\beta9_{\beta+1}\cdots$ and $0.a_1a_2\cdots(a_{\beta-1}+1)0_\beta0_{\beta+1}\cdots$, where $a_{\beta-1}\neq 9$ (the ambiguous notation does not arise if β is a limiting ordinal). The author shows that the set $C_{\omega\epsilon}$ of such sequences is continuous in the sense of Dedekind. The cardinal number of the set $C_{\omega\epsilon}$ is $2^{|\omega\epsilon|}$. The question of the similarity of the sets $C_{\omega\epsilon}$ is investigated and partially answered: $C_{\omega\alpha}$ and $C_{\omega\beta}$, $\alpha\neq\beta$, are dissimilar if $\alpha, \beta\leqq\omega 2$. Also, C_η and C_ξ are dissimilar if $\eta<\omega_\alpha\leqq\xi$, where ω_α is a limiting ordinal. The following question is also investigated: If $C_{\omega\epsilon}=C_1+C_2=C_1'+C_2'$ are two cuts, is C_1 similar to C_1'? If $\epsilon\geqq\omega_1$, or if $1<\epsilon<\omega_1$ and ϵ is non-limiting, the answer is negative. If $\omega\epsilon$ is indecomposable and $<\omega_1$, the answer is affirmative. *J. V. Wehausen.*

Cuesta Dutari, N. Construction of an ordered dense set which is not continuous and whose cardinal is $|\omega_\alpha|$. Revista Mat. Hisp.-Amer. (4) 3, 38–40 (1943). (Spanish) [MF 8354]

Let C be the set of all transfinite sequences $1a_1a_2\cdots a_\beta\cdots$, $\beta<\omega_\alpha$, in which a finite number of the a_β equal 1, and the rest equal 2. Ordered lexicographically, C is a well-ordered

set of cardinal number $|\omega_\alpha|$. The author shows that there are no jumps but are gaps, that is, between any two elements of C there is another element of C, but it is possible to write $C=C_1+C_2$, $C_1<C_2$, such that C_1 does not have a last element and C_2 does not have a first element.

J. V. Wehausen (Columbia, Mo.).

Maximoff, Isaiah. On the continuum hypothesis. Ann. of Math. (2) **44**, 90–92 (1943). [MF 8077]

In an earlier paper [Ann. of Math. (2) **41**, 321–327 (1940); these Rev. **1**, 206] the author considered the space $E_s^{(r)}$ of all sequences

$$x=\{x_0, x_1, \cdots, x_\alpha, \cdots\}, \qquad \alpha\leqq\beta<\Omega_r,$$
$$x=\{x_0, x_1, \cdots x_\alpha\}, \qquad \alpha<\Omega_r,$$

where x_α is any one of the numbers $1, 2, \cdots, \gamma, \cdots, \gamma<\Omega_r$, defined for this space rational and irrational points, established an order relation among points, and proved the space to be a continuum of order $2^{\aleph_r}$. In this paper he first defines intervals in the space, limit ($\mathfrak{Z}$) points for sets of the space, continuity ($\mathfrak{Z}$) for functions with domain and range both in the space, and, moreover, calls a function f hypermeasurable ($\mathfrak{Z}$) in a set $E\subset E_s^r$ if E contains a perfect set in which f is continuous ($\mathfrak{Z}$). He then considers the space E_s^{rk} of sequences of the form

$$x=\{x_0, x_1, \cdots, x_\alpha, \cdots\}, \qquad \alpha<\Omega_{r+k},$$

with x_α a point of E_s^r. Continuity ($\mathfrak{Z}$) and hypermeasurability ($\mathfrak{Z}$) of a function with domain and range in E_s^{rk} is defined by means of the continuity ($\mathfrak{Z}$) and hypermeasurability ($\mathfrak{Z}$) of components of this function. Two specific sets E_1 and E_2 of E_s^{rk} are defined and a function f is said to be a function of N. Parfentieff if f is on E_2 to E_s^{rk}, if E_1 is not the range of f, and if f is regular on E_2, i.e., $(x_1\epsilon E_2)(x_2\epsilon E_2)(x_1\neq x_2)$ implies $f(x_1)\neq f(x_2)$. The main results of the paper are given in two theorems. Theorem 1: if a function f is continuous ($\mathfrak{Z}$) and regular in E_2, then f is a function of N. Parfentieff. Theorem 2: every hypermeasurable ($\mathfrak{Z}$) function which is regular in E_2 is a function of N. Parfentieff.

J. F. Randolph (Ithaca, N. Y.).

Maximoff, Isaiah. On functions of class 1 having the property of Darboux. Amer. J. Math. **65**, 161–170 (1943). [MF 7783]

A necessary and sufficient condition that a function of class 1 have the property of Darboux on $[a, b]$ is obtained which has the same relation to Lebesgue's characterization of functions of class 1 that the author's previous condition [Prace Mat.-Fiz. **43**, 241–265 (1936), in particular, pp. 241–242, 258–259] has to Baire's characterization of functions of class 1. Also a necessary condition that there exist a finite function φ of class 1 having the property of Darboux and satisfying the functional equation $g(x)=f[\varphi(x)]$ is found.

J. F. Randolph (Ithaca, N. Y.).

Poulkine, S. Sur l'itération des fonctions d'une variable indépendante. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **6**, 71–108 (1942). (Russian. French summary) [MF 7697]

Let $y=f(x)$ be a function defined for $-\infty<x<+\infty$ and subject to certain conditions. Let x_0 be a fixed number. The author considers geometric properties of the iterations $f(x_0)$, $f[f(x_0)]$, $\cdots$. The results cannot be stated in a simple way.

A. Zygmund (South Hadley, Mass.).

Federer, H. and Morse, A. P. Some properties of measurable functions. Bull. Amer. Math. Soc. **49**, 270–277 (1943). [MF 8223]

Let I denote a fixed closed interval and f a numerically valued measurable function defined on I. For any set $X\subset I$ let $f(X)$ denote the functional image of X, and for any point $y\epsilon f(I)$ let y^* denote the set $E_x[f(x)=y]$. Define $\mathfrak{K}=E_y[y^*$ is infinite$]$, $\mathfrak{Q}=E_y[y^*$ is nondenumerable$]$. The following theorems are proved. (1) If A is a measurable subset of I then there is a measurable subset B of A such that $f(B)=f(A)$ and f assumes each value $y\epsilon f(A)$ exactly once on B. (2) There is a measurable set C such that $f(C)=\mathfrak{K}$ and y^*-C is finite for each $y\epsilon f(I)$. (3) If $\epsilon>0$ there is a measurable set $L\subset I$ with $f(L)=\mathfrak{K}$ and $|L|<\epsilon$; there is a measurable set $R\subset I$ with $f(R)=\mathfrak{Q}$ and $|R|=0$. From these follow almost immediately two theorems of Banach, namely, N implies T_2 and S is equivalent to $N\cdot T_1$ [notation as in Saks, Theory of the Integral, Warsaw, 1937, pp. 224, 277, 282]. Finally it is indicated under what conditions these results can be extended to functions on one metric space to another.

R. L. Jeffery (Wolfville, N. S.).

Reichelderfer, Paul V. On bounded variation and absolute continuity for parametric representations of continuous surfaces. Trans. Amer. Math. Soc. **53**, 251–291 (1943). [MF 8123]

In a preceding paper in conjunction with Radó [Trans. Amer. Math. Soc. **49**, 258–307 (1941); these Rev. **2**, 257], the writer has defined the following notions with respect to continuous transformations $T: x=x(u, v)$, $y=y(u, v)$ defined on bounded domains $\mathfrak{D}$ in the (u, v) plane: (1) base sets for T, (2) the bounded variation and absolute continuity of T with respect to a base set B in $\mathfrak{D}$ (T being then BVB or ACB), (3) a maximal model continuum in $\mathfrak{D}$ corresponding to a point P in the (x, y) plane, (4) the essential multiplicity $K(P, T, \mathfrak{D})$ of a point P in the (x, y) plane, (5) the generalized Jacobian of a transformation T. Many theorems were proved concerning transformations which were $BV\epsilon$ and $AC\epsilon$, where ϵ was the particular base set which consisted of all maximal model continua (corresponding to any point P) which were points.

In this paper, theorems are proved concerning transformations which are BVE or ACE, where E is the base set consisting of all points in $\mathfrak{D}$ which belong to any maximal model continuum in $\mathfrak{D}$. The results are applied to surfaces in the sense of Fréchet. Any transformation T determines a flat surface and with each surface in 3-space are associated three flat projection surfaces on the coordinate planes. For a flat surface S, the number $K(P, T, \mathfrak{D})$ is independent of the representation T of S and the essential variation $eV(S)$ is defined as $\iint K(P, S)dP$ ($+\infty$ if this is not summable). The essential area $eA(S)$ of a general surface is defined in the obvious way in terms of the essential variations of the projections of subsurfaces of S. It is shown that $\mathfrak{G}(S)\leqq eA(S)\leqq A(S)$ and that $eA(S)$ is lower semicontinuous in S, $\mathfrak{G}(S)$ and $A(S)$ denoting the Geöcze and Lebesgue areas of S, respectively. The equality of pairs of these areas is demonstrated under very general conditions. Well-known results for curves are generalized to the following extent: (1) a necessary and sufficient condition that $eA(S)$ be finite is that the essential variations of its projection surfaces all be finite; (2) a necessary condition that the essential variation of a flat surface be finite is that all of its representations be $BV\epsilon$ and a sufficient condition is that at least one repre-

sentation be BVE; (3) if $eA(S)$ is finite then, for each representation, the three generalized Jacobians are summable with $eA(S) \geqq \iint (J_1^2+J_2^2+J_3^2)^{\frac{1}{2}} du dv$. A necessary condition that the equality hold is that the representation be $AC\epsilon$; a sufficient condition is that the representation be ACE. In case a representation is such that the partial derivatives exist almost everywhere, it follows that the ordinary Jacobians are summable; in this case the results above hold with the ordinary Jacobians replacing the generalized Jacobians. In the case of a flat surface, a necessary and sufficient condition for the equality above is that the representation be ACE. A number of other previous results along these lines are extended and many theorems are proved concerning convergence in variation and in area.

C. B. Morrey, Jr. (Aberdeen, Md.).

Theory of Functions of Complex Variables

Pontrjagin, L. On zeros of some transcendental functions. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 6, 115–134 (1942). (Russian. English summary) [MF 7691]

Let $h(z, t)$ be a polynomial of degree r and s in z and t, respectively. If $h(z, t)$ contains this term, this term is the principal one. Theorem 1. If $h(z, t)$ does not contain the principal term, $H(z, e^z)=0$ has an infinity of roots with arbitrarily large real parts. Let $f(z, u, v)$ be a polynomial with real coefficients. Let

$$f(z, u, v)=\sum_{m, n} z^m \phi_m^{(n)}(u, v),$$

where $\phi_m^{(n)}$ is homogeneous in u, v of degree n. Let r be the degree of f with respect to z, and s the degree with respect to both u and v; $z^r\phi_r^{(s)}(u, v)$ is the principal term of f. Theorem 2. If $f(z, u, v)$ does not contain the principal term, then $f(z, \cos z, \sin z)=0$ has an infinity of roots with arbitrarily large imaginary parts. Other theorems concern the distribution of the zeros of $H(z, e^z)$, $f(z, \cos z, \sin z)$ when the corresponding principal terms do exist. For instance, theorem 3 states that $f(z, \cos z, \sin z)=0$ has $4ks+r$ zeros in the region $-2k\pi+\epsilon \leqq z \leqq 2k\pi+\epsilon$ if k is sufficiently large. A necessary and sufficient condition that all the preceding roots be real is that there exist $2ks+r$ real zeros in the interval $[-2k\pi+\epsilon, 2k\pi+\epsilon]$ for k sufficiently large.

S. Mandelbrojt (Houston, Tex.).

Tschebotaröw, N. On entire functions with real interlacing roots. C. R. (Doklady) Acad. Sci. URSS (N.S.) 35, 195–197 (1942). [MF 7616]

If $g(z)$ and $h(z)$ are two polynomials, then the following two properties are equivalent: (1) $g(z)$ and $h(z)$ have only real roots which are interlacing; (2) for any λ, μ the polynomials $\lambda g(z)+\mu h(z)$ have only real roots. If $g(z)$, $h(z)$ are entire functions satisfying (2), they are said to form a real couple. For entire functions this property is narrower than property (1). If $g(z)$, $h(z)$ form a real couple, all their roots are real, simple and interlacing. The author proves also that, if $g(z)$, $h(z)$ form a real couple, then from $\mathfrak{J}(z_1)\mathfrak{J}(z_2)>0$ it follows that

$$\mathfrak{J}(g(z_1)/h(z_1))\mathfrak{J}(g(z_2)/h(z_2))>0,$$

where $\mathfrak{J}(z)$ is the imaginary part of z. *S. Mandelbrojt.*

Hughes, H. K. On a theorem of Newsom. Bull. Amer. Math. Soc. 49, 288–292 (1943). [MF 8226]

The following theorem is established concerning the entire function $f(z)$ defined by the power series

$$(1) \qquad f(z)=\sum_{n=0}^{\infty} g(n)z^n.$$

Let $g(w)$ $(w=x+iy)$ be single-valued and analytic in the finite w-plane except for a singularity at $w=w_1 \neq$ negative integer, and for all $|x|$ and $|y|$ sufficiently large let

$$(2) \qquad |g(x+iy)|<Ke^{k\pi|y|},$$

where K, k are positive constants, k being an integer. Then

$$(3) \quad f(z)=\int_{-l-\frac{1}{2}}^{\infty} g(x)[(-1)^{k+1}\cdot z]^x \cdot \frac{\sin k\pi x}{\sin \pi x}dx - \sum_{m=1}^{l}\frac{g(-m)}{z^m}-\frac{1}{2i}\int_C \frac{g(w)z^w}{e^{k\pi i w}\sin \pi w}dw+\xi(z, l),$$

where $\lim z^l\xi(z, l)=0$ as $|z|\to\infty$, for all l, in the range $|\arg[(-1)^{k+1}\cdot z]|<\pi$, and where C is a loop surrounding $w=w_1$ and extending to infinity in any direction lying in either the third or fourth quadrant.

If the hypothesis is restricted so that $g(w)$ is single-valued and analytic in the finite w-plane without exception, and satisfies condition (2) there, then Newsom showed [Amer. J. Math. 60, 561–572 (1938)] that (3) holds with, however, the loop integral removed. Putting back the singularity at $w=w_1$, Newsom [loc. cit.] obtained (3) with a more complicated loop integral than the one here given; the work of Hughes consists in replacing Newsom's integral by the simpler one given in (3). Relation (3) undergoes simple modification if the singularity w_1 is a negative integer or if more than one singularity is permitted. The result of Hughes has been utilized by C. G. Fry and Hughes [Duke Math. J. 9, 791–802 (1942); these Rev. 4, 137] to obtain asymptotic developments of certain classes of power series.

Two typographical errors should be noted. In relation (2), the exponent of e should be $k\pi|y|$ instead of $\pi|y|$; and in relation (3), the summation should be from $m=1$ to l instead of $m=0$ to l. *I. M. Sheffer* (State College, Pa.).

Shah, S. M. On the relations between the lower order and the exponent of convergence of zeros of an integral function. J. Univ. Bombay 11, 10–13 (1942). [MF 8241]

Examples are constructed to illustrate the relationships which are possible among the order, lower order, exponent of convergence of the zeros and lower order of the zeros for an entire function. [Cf. the author's earlier paper, J. Indian Math. Soc. (N.S.) 6, 63–68 (1942); these Rev. 4, 137.]

E. S. Pondiczery (Princeton, N. J.).

Shah, S. M. On integral functions of infinite order. J. Univ. Bombay 11, 9 (1942). [MF 8182]

Examples are given [proofs omitted] for an extension to entire functions of infinite order of the author's previous results for functions of finite order [Bull. Amer. Math. Soc. 46, 909–912 (1940); these Rev. 2, 183]. Let $\psi(x)$ be a positive nondecreasing function for which $\log x=o(\log \psi(x))$ as $x\to\infty$. There exists an entire function of infinite order for which (with the usual notation) $\limsup T(r)/\{\psi(r)n(r)\}=\infty$ and another for which $\liminf \psi(r)T(r)/n(r)=0$.

E. S. Pondiczery (Princeton, N. J.).

Kronsbein, J. Analytical expressions for some extremal schlicht functions. J. London Math. Soc. 17, 152–157 (1942). [MF 8262]

The author expresses in terms of elliptic functions the extremal functions corresponding to extremal problems of the conformal mapping of multiply-connected regions which were considered by Grötsch [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 80, 367–376, 497–502 (1928).] The classical results of Koebe are shown to be limiting cases of the present ones. Application of these results to the solution of certain electrostatic problems is indicated.

M. H. Heins (Chicago, Ill.).

Seidel, W. and Walsh, J. L. On the derivatives of functions analytic in the unit circle and their radii of univalence and of p-valence. Trans. Amer. Math. Soc. 52, 128–216 (1942). [MF 6998]

The main object of the paper is the study of the behavior of the expression $f^{(k)}(z)(1-|z|)^k$ as $|z|\to 1$ for a function $f(z)$ which is analytic in $|z|<1$ and belongs to one of the following classes: (i) $f(z)$ is univalent, (ii) $f(z)$ is bounded, (iii) $f(z)$ omits two values in $|z|<1$. Let $f(z)$ be analytic in $|z|<1$ and let R denote the Riemann configuration over the w-plane onto which $f(z)$ maps $|z|<1$. Let w_0 be a point of R. Then the radius $D_1(w_0)$ of the largest smooth (open) circle whose center is at w_0 and which is wholly contained in R is called the radius of univalence of R at w_0. With this definition the authors prove the following. If $f(z)$ is regular and univalent in $|z|<1$ and if $w_0=f(z_0)$ ($|z_0|<1$), then

$$(1)\qquad D_1(w_0)\leqq |f'(z_0)|(1-|z_0|^2)\leqq 4D_1(w_0).$$

(Both inequalities are sharp.) Hence, if $|z_n|<1$ ($n=1, 2, \cdots$) and $w_n=f(z_n)$, a necessary and sufficient condition that

$$(2)\qquad \lim_{n\to\infty} f'(z_n)(1-|z_n|)=0$$

is that

$$(3)\qquad \lim_{n\to\infty} D_1(w_n)=0,$$

and $|f'(z_n)|(1-|z_n|)$ is bounded if and only if $D_1(w_n)$ is bounded. Several other corollaries are derived from (1). The analogous result for functions of class (ii) is the following. If $f(z)$ is regular and $|f(z)|<M$ for $|z|<1$, then

$$(4)\quad |D_1(w_0)|\leqq |f'(z_0)|(1-|z_0|^2)\leqq [8MD_1(w_0)]^{\frac{1}{2}},\ w_0=f(z_0),$$

and it follows also for a bounded function that a necessary and sufficient condition for (2) is (3).

For univalent functions the right hand side of (1) is extended to higher derivatives, and it is shown that $\lim_{n\to\infty} D_1(w_n)=0$ implies that

$$\lim_{n\to\infty} f^{(k)}(z_n)(1-|z_n|)^k=0$$

(for any $k=1, 2, \cdots$). To solve the (more difficult) analogous problem for functions of class (ii) and (iii), the authors introduce the radius of p-valence of a Riemann surface at a point w_0. Let R be a Riemann configuration over the w-plane; let w_0 be a point of R, not a branch point of order greater than $p-1$ ($p>1$). Suppose that C_p is a simply connected region on R which contains w_0, lies over the circle $|w-w_0|<\rho$ and covers it precisely p times; C_p is called a p-sheeted circle of center w_0 and radius ρ. The radius of p-valence $D_p(w_0)$ of R at w_0 is now defined as follows: (a) for $p=1$, $D_p(w_0)=D_1(w_0)$; (b) if w_0 is a branch point of order greater than $p-1$, then $D_p(w_0)=0$; (c) for any other point w_0, $D_p(w_0)$ is the radius of the largest p-sheeted circle with center w_0 contained in R, if such a circle exists, and is otherwise $D_{p-1}(w_0)$. Among the properties of $D_p(w_0)$, the following is the most essential one. Let $\{f_n(z)\}$ be a sequence of functions analytic in $|z|<1$ and converging uniformly in every closed subregion of $|z|<1$ to an analytic function $f(z)$. Let $|z_0|<1$ and let $w_n=f_n(z_0)$, $w_0=f(z_0)$. Let $D_p(w_n)$ and $D_p(w_0)$ pertain to the images of $|z|<1$ by the maps $w=f_n(z)$ and $w=f(z)$, respectively. Then $\lim_{n\to\infty} D_p(w_n)=D_p(w_0)$. This theorem is now used to prove the desired extension. If $f(z)$ is regular and bounded in $|z|<1$, and if $|z_n|<1$ and $w_n=f(z_n)$, then

$$\lim_{n\to\infty} f^{(k)}(z_n)(1-|z_n|)^k=0,\ k=1, 2, 3, \cdots, p,$$

if and only if $\lim_{n\to\infty} D_p(w_n)=0$. This result is also derived from the following analogue of (4). If $f(z)$ is regular and $|f(z)|<M$ in $|z|<1$, then there exist two constants λ_p, depending only on p, and Λ_p, depending on p and M, such that

$$\lambda_p D_p(w_n)\leqq \sum_{k=1}^{p}\left|\sum_{\nu=0}^{k-1}(-1)^k C_{k-1,\nu}\bar{z}_n{}^{\nu}\frac{(1-|z_n|^2)^{k-\nu}f^{(k-\nu)}(z_n)}{(k-\nu)!}\right| \leqq \Lambda_p[D_p(w_n)]^{2^{-p}}.$$

Schottky's theorem is now used to extend both of these results to functions of class (iii), under the assumption that the sequence $w_n=f(z_n)$ is bounded. If $|w_n|\to\infty$ as $n\to\infty$, these theorems are not true, as is shown by examples, and are replaced by other results obtained in this case.

The last chapter of the paper contains numerous applications to the general subject of limit values of analytic functions at a boundary point related to theorems of Lindelöf and Montel. Further applications include an extension of Bloch's theorem, and some extensions of the above quoted inequalities to other classes of functions defined in $|z|<1$, such as p-valent analytic functions and meromorphic functions which omit three values.

S. E. Warschawski.

Schiffer, Menahem. Variation of the Green function and theory of the p-valued functions. Amer. J. Math. 65, 341–360 (1943). [MF 8208]

Let $z=f(\zeta)=\zeta+a_2\zeta^2+a_3\zeta^3+\cdots$ be regular and schlicht in $|\zeta|<1$. It is well known that the family of all such functions is compact, and so for any fixed $n>1$ there are functions of the family with nth coefficient of maximum absolute value. Suppose that f has maximal nth coefficient, which may without loss of generality be taken real and positive, and write

$$P_n(x)=\sum_{\nu=2}^{n} a_n{}^{(\nu)}x^{\nu-1},$$

where $a_n{}^{(\nu)}$ is the nth coefficient of $\{f(\zeta)\}^\nu$. In an earlier paper [Proc. London Math. Soc. (2) 44, 432–449 (1938)] the author showed by a variational method that the map of $|\zeta|<1$ by f is a slit domain whose boundary satisfies the differential equation

$$(1)\qquad \frac{1}{a_n}\left(\frac{z'(t)}{z(t)}\right)^2 P_n\left(\frac{1}{z(t)}\right)+1=0,$$

where t is a suitably chosen real parameter. In this paper he devises a new variational method and obtains a differential equation for f, namely,

$$(2)\quad \zeta^2\frac{f'(\zeta)^2}{f(\zeta)^2}P_n\left(\frac{1}{f(\zeta)}\right)=(n-1)a_n+\sum_{\nu=1}^{n-1}(\nu a_\nu\zeta^{-n+\nu}+\nu\bar{a}_\nu\zeta^{n-\nu}).$$

[Equation (2) has been obtained independently by the

reviewers who (in a paper to appear in the Duke Math. J.) derive from it a new proof of Löwner's well-known result that $|a_3|\leqq 3$.] The author's method, which is of interest in itself, may be briefly described as follows. Let Z be the closed z-plane, and let $z^*=z+\rho z/(z-z_0)$, where ρ is a small complex number. If we remove from Z the interior k_0 of a small circle with center at z_0 and choose $|\rho|$ sufficiently small, then $z^*(z)$ is schlicht in the remaining region and maps it into the plane with a hole k_0^* cut out. Let D be the map of $|\zeta|<1$ by f, and let z_0 as well as k_0 be interior to D. The function $z^*(z)$ transforms $D-k_0$ into a doubly-connected region $D^*-k_0^*$ and, adding to the latter region the interior of k_0^*, we obtain a simply-connected domain D^*. Now let $g(z,x)$ and $g^*(z,x)$ be the respective Green's functions of D and D^*, and let $f^*(\zeta)$ be the function mapping $|\zeta|<1$ onto D^* with $f^*(0)=0$, $f^{*\prime}(0)>0$. Using Green's theorem the difference $g^*(z,x)-g(z,x)$ is calculated to within an error $O(|\rho|^2)$, from which the difference $f^*(\zeta)-f(\zeta)$ is easily determined to the same degree of accuracy. Denoting the coefficients of $f^*(\zeta)$ by a_ν^*, equation (2) results from the inequality $|a_n^*/a_1^*|\leqq a_n$, which is true for all small ρ.

The author next shows that (2) can be deduced from (1), the derivation being ingeniously affected by means of an interesting relation between $P_n(x)$ and the nth Faber polynomial of f. He also shows that (1) is a consequence of (2), but here the reviewers do not follow the argument. Finally it is shown that the method of variations is applicable to the family of functions $z=f(\zeta)=\zeta+a_2\zeta^2+a_3\zeta^3+\cdots$ which are regular and p-valent in $|\zeta|<1$ and assume the value zero only at the origin. Each function of the family maps $|\zeta|<1$ on a domain D spread over a closed p-sheeted Riemann surface R in the z-plane. Since the subclass F_g of all functions of the family whose domains D can be spread over p-sheeted Riemann surfaces R with genera not exceeding a fixed number g is compact, it is sufficient to consider the coefficient problem for F_g. It is shown that a function f of F_g with maximum a_n $(a_n>0)$ satisfies the differential equation

$$(3)\quad \frac{f(\zeta)}{\zeta^2 f'(\zeta)^2}\left\{(n-1)a_n+\sum_{\nu=1}^{n-1}(\nu a_\nu\zeta^{-n+\nu}+\nu\bar{a}_\nu\zeta^{n-\nu})\right\}$$
$$+\sum_{\nu=2}^{n}\frac{a_n^{(\nu)}}{\nu-1}\tau'_{\nu-1}(f(\zeta),f(0))=w'(f(\zeta)),$$

where w is an integral of the first kind on R and $\tau_{\nu-1}(z,f(0))$ is a normal integral having everywhere on R a finite and uniform derivative with respect to the local parameter with the exception of the point $f(0)$, in the neighborhood of which

$$\tau_{\nu-1}(z,f(0))=(1/z^{\nu-1})+\alpha+\beta z+\cdots.$$

The reviewers do not understand what is meant by the statement "In the special case where Δ is simply-connected and, therefore, can be supposed to be the unit circle . . ." on page 341. *A. C. Schaeffer and D. C. Spencer.*

Macintyre, A. J. and Wilson, R. The logarithmic derivatives and flat regions of analytic functions. Proc. London Math. Soc. (2) **47**, 404–435 (1942). [MF 8269]

Using the Nevanlinna potential-theory, the authors extend results of P. Boutroux [Acta Math. **28**, 97–224 (1904)] on the asymptotic behaviour of the logarithm of an integral function and its derivatives. For a meromorphic function $F(z)$, $h>0$, $k>1$, they obtain

$$\left|\frac{F'(z)}{F(z)}\right|<\frac{KT(kr)}{r},\quad |D^2\log F(z)|<\frac{KT(kr)\log T(kr)}{r^2},$$
$$|D^q\log F(z)|<\frac{KT(kr)^{q/2}}{r^q},\qquad q>2;\quad K=K(h,k),$$

for all $|z|$ in $|z|<r$ (r large), except for a set of at most $KT(kr)$ circles, the sum of whose areas is at most $h^2\pi r^2$. Somewhat weaker inequalities of the same type hold when the exceptional set of circles has the sum of its radii at most hr. Here $T(r)$ denotes Nevanlinna's characteristic function. For meromorphic functions $F(z)$ of finite order whose poles are of permanent defect there exist arbitrarily large values of z such that

$$|\log F(z)-\log F(z_1)|<A(T(r)/r)(z-z_1)\leqq AC[T(r)]^{\frac{1}{2}},$$
$$|\log F(z_1)|>BT(r),$$

where $|z_1|=r$ and z is any point in $|z-z_1|<Cr/[T(r)]^{\frac{1}{2}}$ and A, B and C are positive constants depending only on the order of $F(z)$ and the defect of the poles.

The authors extend these results to functions regular or meromorphic in an angle. Of the numerous results in this direction a typical one may be mentioned. Let $F(z)$ be regular and, in the half-plane $I(z)\geqq 0$, $\log|F(z)|\leqq H|z|^\rho$ $(\rho>0,\ |z|>r_0)$; then, if $\rho>1$,

$$|F'(iy)/F(iy)|<KR^{\rho-1}$$

for the majority of values y in $0<y<R$ for all sufficiently large R. If $\rho=1$ the right-hand side of the inequality is replaced by $K\log R$, and if $\rho<1$ by K. In this last case

$$|(F'(iy)/F(iy))-i\eta|<\epsilon,$$

η being a positive constant depending upon the function.

M. S. Robertson (New Brunswick, N. J.).

Weyl, Joachim. Exponential curves. Duke Math. J. **10**, 123–143 (1943). [MF 8106]

The principal concern of this paper is to illustrate the general theory of meromorphic and analytic curves initiated by the author in collaboration with H. Weyl by means of the theory of exponential curves. The investigations of the present paper are related to the work of Pólya [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1920**, 285–290 (1920)] and Schwengeler [Thesis, Zürich, 1925] on finite exponential sums. Two classes of curves are considered: the normal exponential curves defined by

$$x_i=x_i(z)=\exp(\lambda_i z),\qquad (i=0,1,\cdots,k)$$

in the complex k-dimensional space

$$\mathfrak{R}:[x_1/x_0,\,x_2/x_0,\,\cdots,\,x_k/x_0],$$

and the general exponential curves defined in an n-dimensional subspace $\mathfrak{R}'$: $[w_1/w_0,\cdots,w_n/w_0]$ of $\mathfrak{R}$ by

$$w_i=w_i(z)=\sum_0^k a_{i\nu}\exp(\lambda_\nu z),\quad (i=0,1,\cdots,n\leqq k).$$

The study is carried out in terms of an indicator diagram, the smallest convex polygon in the complex plane containing the conjugates of the λ_i $(i=0,1,\cdots,k)$.

Two auxiliary theorems are introduced. The first states that, if

$$Q(z)=\sum_0^k|\exp(\lambda_\nu z)|^2,$$

then

$$\int_0^{2\pi}\log Q(re^{i\varphi})d\varphi-2rU=O(1),$$

where U is the circumference of the indicator diagram. The second states that, if

$$H=\sum_{\rho,\sigma} c_{\rho\sigma}\bar{x}_\rho x_\sigma \geqq 0, \quad H^0=\sum_{\rho,\sigma} c^0_{\rho,\sigma}\bar{x}_\rho x_\sigma \geqq 0,$$

where H and H^0 are Hermitian forms ($\rho, \sigma=0, \cdots, k$) with c_{ii} and c^0_{ii} ($i=0, 1, \cdots, k$) different from zero, and if

$$H(z)\equiv\sum_{\rho,\sigma} c_{\rho\sigma} \exp(\bar{\lambda}_\rho\bar{z}) \exp(\lambda_\sigma z),$$

$$H^0(z)\equiv\sum_{\rho,\sigma} c^0_{\rho\sigma} \exp(\bar{\lambda}_\rho\bar{z}) \exp(\lambda_\sigma z),$$

then

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\log\frac{H(re^{i\varphi})}{H^0(re^{i\varphi})}d\varphi\right\}=\mathfrak{A}\left\{\log\frac{c_{ii}}{c^0_{ii}}\right\},$$

where $\mathfrak{A}$ denotes the arithmetical mean.

The fundamental quantities of the theory of meromorphic curves are then studied for the special classes of curves considered. Typical of the results obtained are:

$$(1)\qquad T(r)-\frac{r}{2\pi}U=O(1),$$

$$(2)\qquad \lim_{r\to\infty} m(r, \alpha)=\mathfrak{A}(\log \alpha_i^{-1}),$$

the notation being the standard one for the general theory.

M. H. Heins (Chicago, Ill.).

Manel, Bella. The conformal mapping of multiply-connected domains on the basis of Plateau's problem. Univ. Nac. Tucumán. Revista A. 3, 141–149 (1942). [MF 8149]

Methods used for the solution of the problem of Plateau have been adapted by Courant [Duke Math. J. 5, 814–823 (1939); Univ. Nac. Tucumán. Revista A. 2, 141–149 (1941); these Rev. 1, 111; 4, 9] and his students [Courant, Manel and Shiffman, Proc. Nat. Acad. Sci. U. S. A. 26, 503–507 (1940); Shiffman, ibid., 27, 137–139 (1941); these Rev. 2, 84, 186] to the study of conformal mapping of multiply-connected domains on members of simple classes of normal domains. The point of departure is the fact that a variational problem involving the Dirichlet functional has been solved. The development accordingly is shorter than the presentation of Koebe [Acta Math. 41, 305–344 (1918)] for similar normalizations. It is now shown that the class of plane Jordan domains D of connectivity k is conformally equivalent to a class of normal domains B_1 bounded by the real axis and $k-1$ radial slits, and also to a class of normal domains B_2 bounded by the unit circle and $k-1$ concentric circular arcs.

E. F. Beckenbach (Austin, Tex.).

Fourier Series and Generalizations, Integral Transforms

Salem, R. On some singular monotonic functions which are strictly increasing. Trans. Amer. Math. Soc. 53, 427–439 (1943). [MF 8417]

The author investigates properties, in particular, the modulus of continuity and the order of the Fourier-Stieltjes coefficients, of two singular and strictly increasing functions. One of these functions is introduced by the author, mainly to give a direct and simple construction of a singular strictly increasing function [the known examples of such functions are usually either not very simple, or their construction is not direct]. The second function is the function $\phi(x)$ of Minkowski [see his Gesammelte Abhandlungen, vol. 2, Leipzig, 1911, pp. 50–51], later studied by Denjoy [J. Math. Pures Appl. (9) 17, 105–151 (1938)]. The author shows that $\phi(x)\epsilon$ Lip α, where $\alpha=\frac{1}{2}\log 2/\log\theta$, and θ is the Fibonacci number $\frac{1}{2}(5^{\frac{1}{2}}+1)$.

A. Zygmund.

Wang, Fu Traing. Note on the absolute summability of trigonometrical series. J. London Math. Soc. 17, 133–136 (1942). [MF 8258]

The author has previously shown [same J. 16, 174–176 (1941); these Rev. 3, 231] that, if $\epsilon>0$ and if

$$\sum(a_n^2+b_n^2)(\log n)^{1+\epsilon}<\infty,$$

then the series

$$(*)\qquad \sum(a_n \cos nx+b_n \sin nx)$$

is summable $|C, \alpha|$, $\alpha>\frac{1}{2}$, almost everywhere. He now shows that, if $\epsilon=0$, then at almost every point x the series (*) may even be nonsummable $|A|$.

A. Zygmund.

Reves, George E. and Szász, Otto. Some theorems on double trigonometric series. Duke Math. J. 9, 693–705 (1942). [MF 7923]

This paper generalizes to trigonometric series of two variables (1) the Cantor-Lebesgue theorem, (2) the Denjoy-Lusin theorem on absolute convergence and (3) two results of Szász proved earlier [Trans. Amer. Math. Soc. 42, 366–395 (1937)]. The following theorem is a consequence of this last generalization. If $f(x, y)$ belongs to Lip (α, β) and is of bounded variation H, the sum of the kth powers of the absolute value of the Fourier coefficients of $f(x, y)$ converges for

$$k>\max\left(\frac{2}{2+\alpha}, \frac{2}{2+\beta}\right).$$

R. Salem (Cambridge, Mass.).

Wintner, Aurel. Riemann's hypothesis and harmonic analysis. Duke Math. J. 10, 99–105 (1943). [MF 8104]

Let $f(t)$ be defined in $0\leqq t<\infty$ and L-integrable in every bounded subinterval; $f(t)$ is said to have a Fourier expansion if

$$M(f_\lambda)=\lim_{T\to\infty} T^{-1}\int_0^T f(t)e^{i\lambda t}dt$$

exists for every λ and is 0 except for an enumerable set of λ's. It is not known whether a function $f(t)$ exists such that $M(f_\lambda)$ exists for every λ and is different from 0 for a non-enumerable set of λ's. A function possessing a Fourier development need not be almost periodic. Put

$$S(t)=\pi^{-1}\arg\zeta(\tfrac{1}{2}+it).$$

Then $S(t)$ has the following Fourier expansion:

$$2\pi\varphi(t)\sim\sum_{n=2}^{\infty}\frac{-\Lambda(n)}{n^{\frac{1}{2}}\log(n^{\frac{1}{2}})}\sin(t\log n),$$

where $\Lambda(n)=\log n$ for $n=p^k$ and $\Lambda(n)=0$ for $n\neq p^k$, but $S(t)$ is not almost periodic (B). Write $\varphi_1(t)=\int_0^t\varphi(u)du$. Then

$$\varphi_1(t)\sim\frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\frac{1}{2}}\log^2 n}\cos(t\log n).$$

$S(t)$ is almost periodic (B^2).

P. Erdös.

Sirvint, G. Quelques exemples de séries de Dirichlet dont la suite d'exposants est condensée. Rec. Math. [Mat. Sbornik] N.S. **10**(52), 59–66 (1942). (French. Russian summary) [MF 7769]

It follows from a theorem of V. Bernstein that, if the sequence $\{\lambda_n\}$ is of "finite maximum density," then $H=0$, where $\sigma=H$ is the abscissa of holomorphy of (1) $\sum a_n e^{-\lambda_n s}$ and $\sigma=0$ the abscissa of over-convergence of the same Dirichlet series; C denotes the abscissa of ordinary convergence. It is easy to construct examples of (1) with $\{\lambda_n\}$ of "infinite maximum density" such that $H<0=C$. The author proves that S. Rios' [Bol. Sem. Mat. **4**, 167–174 (1937)] attempt to construct an example with $H<0<C$ is unsuccessful. He gives a theorem which allows such constructions. *S. Mandelbrojt* (Houston, Tex.).

Luntz, G. Sur quelques généralisations des séries de Dirichlet. Rec. Math. [Mat. Sbornik] N.S. **10**(52), 33–50 (1942). (Russian. French summary) [MF 7767]

The author considers series (1) $\sum a_n e^{-\lambda_n s}$ with complex a_n and λ_n, $|\lambda_n|\rightarrow\infty$. He defines, by means of certain functions of two variables which depend on $\{a_n\}$ and $\{\lambda_n\}$, the region of absolute convergence of (1). He also gives theorems concerning absolute convergence of double Laplace-Stieltjes integrals. Furthermore, the author proves that (1) diverges at every point outside of the region of absolute convergence if $\log n=o(|\lambda_n|)$. *S. Mandelbrojt* (Houston, Tex.).

Oberg, Edwin N. On the approximation of functions by sums of orthonormal functions. Bull. Amer. Math. Soc. **49**, 68–80 (1943). [MF 7986]

Let $s_n(x)=\sum_1^n a_k\phi_k(x)$ with given constants a_k and a given system $\{\phi_k(x)\}$ in the interval (a, b). The author obtains bounds for $s_n'(x)$ in the C and L_2 spaces of the form

$$|s_n'(x)|\leqq A_n k(x)\|s_n(x)\|_2,\quad \|\sigma^{\frac{1}{2}}(x)s_n'(x)\|_2\leqq B_n\|s_n(x)\|_2.$$

Estimates of the first type are given for the case in which the $\{\phi_k(x)\}$ are characteristic functions of the set of integral equations

$$\phi(x)=\lambda\int_a^b K(x, t)\psi(t)dt,\quad \psi(x)=\lambda\int_a^b K(t, x)\phi(t)dt.$$

Here $A_n=\lambda_n$ and $k(x)=\|K_x'(x, t)\|_2$. Estimates of the second type are obtained when the functions $\phi_k'(x)$ are orthogonal with respect to a positive weight function $\sigma(x)$. Here B_n is the normalization factor $\|\sigma^{\frac{1}{2}}(x)\phi_n'(x)\|_2$, if these factors form an increasing sequence. Applications to the question of the degree of approximation by sums $s_n(x)$ are given in both cases, and also examples from classical orthogonal polynomials and Bessel functions. The methods are elementary and have little or no contact with methods previously used in special cases by Shohat, McEwen, Hille, Szegö and Tamarkin. *E. Hille* (New Haven, Conn.).

Albert, G. E. The closure of systems of orthogonal functions. Amer. Math. Monthly **50**, 163–169 (1943). [MF 8162]

This paper is primarily expository, and the author states that "the entire procedure may be shown to students having no more preparation than a course in advanced calculus"; in the opinion of the reviewer, however, considerable mathematical maturity on the part of the students would be required if the procedure were to be properly appreciated. The criterion obtained for L^2 closure of the orthonormal set $\{\varphi_n\}$ on (a, b) may be stated as the convergence in mean square of $\sum_{n=0}^\infty\varphi_n(t)\int_a^t\varphi_n(x)dx$ to the value $\frac{1}{2}$, or equivalently as

$$\lim_{n\rightarrow\infty}\int_a^b\{1-2\sigma_n(t)\}^2dt=0,$$

$$\sigma_n(t)=\sum_{r=0}^n\varphi_r(t)\int_a^t\varphi_r(x)dx.$$

This is shown to apply very simply to Legendre functions and to the trigonometric system. *E. S. Pondiczery.*

Functional Analysis, Ergodic Theory

Millsaps, Knox. Abstract polynomials in non-Abelian groups. Bull. Amer. Math. Soc. **49**, 253–257 (1943). [MF 8220]

For "abstract polynomials," with domain in a non-Abelian group and range in an Abelian group, the proof of a decomposition theorem in terms of monomials is outlined. *F. J. Murray* (New York, N. Y.).

Phillips, R. S. On weakly compact subsets of a Banach space. Amer. J. Math. **65**, 108–136 (1943). [MF 7780]

A subset G of a Banach space X is said to be $\aleph_\alpha$-compact if for every directed subset (x_τ) of G of power $\leqq\aleph_\alpha$ there exists an $x_0\epsilon X$ such that $\lim_\tau\inf\bar{x}(x_\tau)\leqq\bar{x}(x_0)\leqq\lim_\tau\sup\bar{x}(x_\tau)$ for every $\bar{x}$ in the adjoint space $\bar{X}$. A weakly closed set G is $\aleph_\alpha$-compact if and only if open coverings of G of power $\leqq\aleph_\alpha$ have finite subcoverings. The paper generalizes a number of known theorems concerning sequentially weak compact sets. For example, the closed convex extension of an $\aleph_\alpha$-compact set is $\aleph_\alpha$-compact. If U is an $\aleph_\alpha$-compact linear operator then so is its adjoint. If the unit sphere in X is $\aleph_\alpha$-compact then the unit sphere in $\bar{X}$ is also and if in addition there is a set of power $\leqq\aleph_\alpha$ total on X then X is reflexive. The theorem of Dunford and Pettis on the representation of weakly compact operators on $L(S)$ is generalized by removing the restriction that S be Euclidean. It is shown that $L^p(X)=L^q(\bar{X})$, $1/p+1/q=1$, providing X is reflexive. *N. Dunford* (New Haven, Conn.).

Bochner, S. and Phillips, R. S. Absolutely convergent Fourier expansions for non-commutative normed rings. Ann. of Math. (2) **43**, 409–418 (1942). [MF 6999]

Let R be a (noncommutative) normed ring, and R' the ring of periodic functions $x(t)=\sum_{-\infty}^\infty a_n e^{int}$, $a_n\epsilon R$, on $0\leqq t<2\pi$ to R, with $\sum\|a_n\|<\infty$. Let the product $x(\cdot)\cdot y(\cdot)$ in R' be the function $x(t)y(t)$. Then the authors show that $x(t)$ has a left inverse in R' if $x(t_0)$ has a left inverse in R for every t_0. This generalizes the well-known theorem of N. Wiener [Ann. of Math. (2) **33**, 1–100 (1932)] from complex numbers to a general R, and is proved by an abstract extension of Wiener's method. A generalization of Wiener's theorem [loc. cit.] on closure of translations in L_1 is obtained similarly. Turning from the method of Wiener to that of I. Gelfand [Rec. Math. [Mat. Sbornik] N.S. **9**(51), 3–24, 51–66 (1941); these Rev. **3**, 51], and letting t be an element of a topological group, the authors obtain stronger and more elaborate theorems. These theorems include extensions of Gelfand's work [loc. cit.], further generalizations of the Wiener reciprocal theorem, generalizations of the work of Wiener and Pitt on reciprocals of Fourier-Stieltjes transforms [Duke Math. J. **4**, 420–436 (1938)] and applications to difference and integral equations. *R. H. Cameron.*

Bailey, R. P. On the convergence of sequences of linear operations. Bull. Amer. Math. Soc. 49, 63–68 (1943). [MF 7985]

A set P of points in a Banach space is called a p-set in case $|x-x^1|\leqq|x+x^1|$ for every pair x, x^1 in P. This is an extension of the notion of positive real numbers. It is shown that if a sequence U_n of continuous linear operators converges strongly at each point of a set dense in some sphere K and if U_nK is, for large n, a p-set then U_nx converges for every x. The conditions of the theorem are also necessary in case $Ux=\lim_n U_nx$ does not vanish identically or in case the space is of finite dimension. An example is given to show that the conditions are not in general necessary in case $Ux\equiv 0$. The theorem constitutes a generalization of previous results of the author [Duke Math. J. 2, 287–303 (1936)].

N. Dunford (New Haven, Conn.).

Pinsker, A. On a class of operations in K-spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) 36, 227–230 (1942). [MF 8091]

A D-operation is an operation between K-spaces (that is, the linear semi-ordered spaces discussed by L. Kantorovitch) having the property that when $\inf(|x_1|, |x_2|)=0$ then $U(x_1+x_2)=U(x_1)+U(x_2)$ and $\inf(|U(x_1)|, |U(x_2)|)=0$. Let X be a separable regular K-space of type K_b^+ and U a D-operation defined on the bounded functions in X and having values in X. Then $\|x_n-x\|\to 0$ implies $U(x_n)\to U(x)(0)$ if and only if $x_n\to x(0)$ implies $U(x_n)\to U(x)(0)$. For this kind of operator a representation of the type $U(x) =\int z(\gamma)de_\gamma(x)$, where $x=\int\gamma de_\gamma(x)$ and $z(\gamma)$ is an X valued function which is uniformly continuous on every finite segment, is given. *N. Dunford* (New Haven, Conn.).

Sirvint, U. Convex sets and linear functionals in an abstract space. I. Convex sets. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 6, 143–170 (1942). (Russian. English summary) [MF 7693]

Sirvint, U. Convex sets and linear functionals in abstract space. II. Linear functionals. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 6, 189–226 (1942). (Russian. English summary) [MF 7593]

These two papers constitute a detailed exposition of results previously announced without proofs in two notes in the Doklady [C. R. Acad. Sci. URSS (N.S.) 26, 119–122, 123–126 (1940); these Rev. 2, 180]. In the second part there is given some additional information concerning linear topological spaces which are not locally convex.

J. V. Wehausen (Columbia, Mo.).

Doob, J. L. and Leibler, R. A. On the spectral analysis of a certain transformation. Amer. J. Math. 65, 263–272 (1943). [MF 8202]

Let Ω^* be the space of sequences $\omega=\{x_n\}$, $n=0, \pm 1, \cdots$, where $0\leqq x_n\leqq 1$, and let P be the measure in Ω^* determined by Lebesgue measure on the interval. Let $U=\exp(2\pi iA)$ be the unitary operator in $L_2(\Omega^*)$ determined by a coordinate shift in Ω^*, that is, $\{x_n\}\to\{x_{n+1}\}$. Then, for the resolution of the identity $E(\lambda)$ determined by A, it is shown that $(E(\lambda)f, g)$ is absolutely continuous after a jump at $\lambda=0$ has been removed. For an arbitrary unitary operator U in $L_2(\Omega)$, where Ω is an arbitrary probability space, it is shown that $(E(\lambda)f, g)$ is absolutely continuous providing f, g lie in the closed linear manifold determined by an orthogonal sequence $\{\phi_j\}$ with $U\phi_j=\phi_{j+1}$. If U is a unitary operator in $L_2(\Omega)$ arising from a measure preserving transformation on Ω then $(E(\lambda)f, g)$ is absolutely continuous after a jump at $\lambda=0$ has been removed providing f, g lie in the closed linear manifold determined by all bounded Baire functions of a finite number of the elements of a sequence $\{\phi_n\}$, where the ϕ_n form an independent set and $U\phi_n=\phi_{n+1}$. Also a condition necessary and sufficient that $\{U^nf\}$ be a sequence of functions independent in pairs is given. *N. Dunford.*

Raikov, D. On absolutely continuous set functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 34, 239–241 (1942). [MF 7606]

Let G be a topological group and $\varphi(E)$ a completely additive set function taking only finite values defined on the least Borel field of sets containing all "strictly open" sets in G; $\varphi(E)$ is "absolutely right-continuous in the sense of Plessner" if $\mathrm{Var}_G|\varphi(Eg)-\varphi(E)|\to 0$ as $g\to e$, the unit element in G; $\varphi(E)$ is right continuous if $|\varphi(Eg)-\varphi(E)|\to 0$ as $g\to e$. If Haar right-invariant measure is defined on G, $\varphi(E)$ is absolutely right-continuous in the sense of Lebesgue if $\varphi(N)=0$ when the right-invariant measure of N is 0. Suitable changes allow one to state left-continuity definitions. Theorem: If Haar measure is defined on G, then right or left continuity in any one sense implies right and left continuity in any sense. *J. V. Wehausen.*

Raikov, D. A new proof of the uniqueness of Haar's measure. C. R. (Doklady) Acad. Sci. URSS (N.S.) 34, 211–212 (1942). [MF 7604]

The uniqueness of Haar's measure on a locally bicompact group is derived as an almost direct consequence of Fubini's theorem. The proof is somewhat similar to that given in Weil, L'integration dans les groupes topologiques et ses applications [Actual. Sci. Ind., no. 869, 1941, pp. 148–9; cf. these Rev. 3, 198]. *J. V. Wehausen.*

Pitt, H. R. Some generalizations of the ergodic theorem. Proc. Cambridge Philos. Soc. 38, 325–343 (1942). [MF 7801]

A new and elegant proof of the Yosida-Kakutani maximal ergodic theorem [Proc. Imp. Acad., Tokyo 15, 165–168 (1939); these Rev. 1, 59] is given (only in the case of finite measure) and the almost everywhere ergodic theorem of G. D. Birkhoff is derived from it very easily without recourse to the mean ergodic theorem. New proofs are also given for N. Wiener's [Duke Math. J. 5, 1–18 (1939)] n-parameter extensions of the mean as well as the almost everywhere theorems. These theorems are shown to hold also when the averages are taken over a class of n-dimensional sets considerably larger than the class of concentric spheres. Applications are given by proving an extension of the well-known theorem of H. Weyl on the approximation of a Riemann integral and by proving a "random" ergodic theorem. *N. Dunford* (New Haven, Conn.).

Pitt, H. R. Random processes in a group. J. London Math. Soc. 17, 88–98 (1942). [MF 7316]

A previous result of the author [same J. 15, 247–257 (1940); these Rev. 2, 231] concerning the existence of a Lebesgue measure is generalized by replacing the real number system by a separable locally compact continuous Abelian group G and the set T of finite sums of intervals by a general field. In brief, it is shown that a Lebesgue measure with certain properties may be defined on the set of all additive G-valued functions defined on T. *N. Dunford.*

Wiener, Norbert and Wintner, Aurel. The discrete chaos. Amer. J. Math. 65, 279–298 (1943). [MF 8204]

The difficulty in finding a mathematical model for classical statistical mechanics is that the system must either be such as to have its state represented in finitely many dimensions or be such as to require infinitely many dimensions. In the former case the chief propositions of thermodynamics are false; in the latter case they are unstatable, until such time as a nontrivial measure in the space has been established, and the difficulties of the theory of measure in function-space are well known. The authors are keenly aware of this dilemma and take it as their incentive for the present paper, which, while not intended to have direct physical application, may serve as a frame in which statistical assumptions not tied down by the assumption of statistical independence may be worked out. The paper undertakes the study of perfectly general spaces, their fields of sets and their product spaces, measure, contingencies, expected values and generating functions, all under assumptions of varying degrees of restrictiveness. An application of Wiener's multi-dimensional ergodic theory is given. The technical complications prevent our further statement of details here. *B. O. Koopman.*

Mathematical Statistics

Wald, Abraham. On a statistical generalization of metric spaces. Proc. Nat. Acad. Sci. U. S. A. 29, 196–197 (1943). [MF 8442]

The relation $F(x; p, q) \dotplus F(x; q, r) \leqq F(x; p, r)$ is proposed as a triangle inequality alternative to those suggested by Karl Menger in his recent statistical generalization of metric spaces [Proc. Nat. Acad. Sci. U. S. A. 28, 535–537 (1942); these Rev. 4, 163]. F is Menger's distribution function and $\dotplus$ stands for composition in the usual statistical sense. The author's inequality enjoys the following advantages: (1) it is specific, (2) "betweenness" has the same properties as in ordinary metric space. *J. L. Vanderslice.*

Siegel, Irving H. Note on a common statistical inequality. J. Amer. Statist. Assoc. 38, 217–222 (1943). [MF 8436]

Basic to the discussion is the relation

$$\Delta = \sum st \sum uv - \sum sv \sum ut \gtreqless 0,$$

all sums having the same index range. For use in applications Δ is exhibited as a matrix product and as a determinant product sum. A further manipulation brings in weighted correlation coefficients whereby Δ becomes a weighted product moment times the square of the weight sum. There follow numerous examples of inequalities arising in statistics to which the previous identities can be applied; for example, the signs of certain index number differences, correlation coefficients and regression line parameters depend on quantities like Δ. Proofs of certain uni-directional inequalities, Cauchy's for example, are facilitated by the previous analysis. In this connection various correlation coefficients, weighted and partial, are expressed in alternative cosine forms bringing in an analogy with spherical trigonometry. *J. L. Vanderslice* (College Park, Md.).

***Treloar, Alan E. Random Sampling Distributions.** Burgess Publishing Co., Minneapolis, Minn., 1942. 94 pp. $2.25.

1943

The book is intended as an introduction to the theory of random sampling distributions and tests of significance for the benefit of students who do not have advanced mathematical training. It contains the following chapters: I: Statistical Bases of Inference; II: The Random Sampling Distribution of Means; III: Random Sampling Differences between Means; IV: Sampling Errors of the Standard Deviation; V: The Estimation of σ from s; VI: Comparison of Standard Deviations; VII: Student's Distribution; VIII: Significant Differences and Student's Distribution; IX: The Analysis of Variance; X: Sampling Errors of the Correlation Coefficient. Many of the "proofs" and "derivations" given in the book hardly deserve to be considered as such; they merely make the results plausible to the reader. However, this may be excusable, since the book is written primarily for readers with little mathematical background. There are a few statements included in the book with which many mathematical statisticians will not quite agree. For instance, the reasons given by the author for the statement on page 64 that the 5 percent level should be used with great caution in the application of the t-test when the number of observations is small will hardly be acceptable to many mathematical statisticians. The statement on page 58 that the statistic given in formula (15) has the t-distribution is unfortunately not correct. On the credit side one may say that several notions of statistical inference, especially that of a test of significance, are very clearly described. The book contains many figures and illustrations which facilitate reading. *A. Wald* (New York, N. Y.).

***Treloar, Alan E. Correlation Analysis.** Burgess Publishing Co., Minneapolis, Minn., 1942. 64 pp. $1.50.

The book gives an outline of a course presented at the University of Minnesota by B. G. Behn and given in former years by the author. A great part of the material included in this book was published previously as part II of "An Outline of Biometric Analysis" issued for the author in 1935 by the present publishers. The book contains the following chapters: I: Differences and Correlation; II: Partial Correlation; III: Multiple Rectilinear Prediction; IV: The Correlation Ratio; V: Curvilinear Regression; VI: The Coefficient of Contingency; VII: Correlation from (2 by m) fold Tables; VIII: Correlation between Ranks. There is very little sampling theory included in the book. While the sampling distributions of the correlation ratio and that of the coefficient of contingency are briefly discussed, there is hardly anything about the distribution of the sample correlation and regression coefficients (total and partial). Population parameters and sample estimates are not very clearly distinguished. Otherwise the book contains a number of formulas which are very useful in correlation and regression analysis. *A. Wald* (New York, N. Y.).

Haavelmo, Trygve. The statistical implications of a system of simultaneous equations. Econometrica 11, 1–12 (1943). [MF 7786]

An important problem in econometrics is to determine the constants in relationships between observable variables. Besides the observable variables and the constants, such relationships often contain "unexplained residuals" which may be considered as stochastic variables. To take a simple formal example, let X and Y be observable quantities, a and b constants, ϵ_1 and ϵ_2 residuals, and let the relationship be described by the system of equations (1) $Y = aX + \epsilon_1$, $X = bY + \epsilon_2$. A procedure frequently used for determining constants would, in this example, consist in applying the method of least squares to each of the equations (1) separately. The author points out that this is unjustifiable if ϵ_1

and ϵ_2 are stochastic variables. The correct formulation of the problem, assuming the probability distribution of ϵ_1, ϵ_2 as known, consists in expressing X and Y as functions of ϵ_1, ϵ_2, finding the joint probability distribution of X, Y and then determining the parameters a, b of this distribution by methods of statistical estimation. The author presents the mathematical model of certain economic processes and uses it to illustrate various aspects of this method.

Z. W. Birnbaum (Seattle, Wash.).

Baker, G. A. Correlations between functions of variables. J. Amer. Statist. Assoc. 37, 537–539 (1942). [MF 7538]

The author points out that it is incorrect to assume that the correlation between x_1 and (x_1+x_2) can never be zero.

Z. W. Birnbaum (Seattle, Wash.).

Davies, G. R. and Bruner, Nancy. A second moment correction for grouping. J. Amer. Statist. Assoc. 38, 63–68 (1943). [MF 8138]

To find adjustments for the second moment calculated from a grouped distribution in case of poor contact, ordinary parabolas are fitted to successive sets of three class frequencies. Such formulae have been found before for the second and higher moments by fitting higher degree parabolas [see, e.g., D. Alter, Ann. Math. Statist. 10, 192–195 (1939)].

C. C. Craig (Ann Arbor, Mich.).

Jones, Howard L. The use of grouped measurements. J. Amer. Statist. Assoc. 36, 525–529 (1941). [MF 8314]

Assuming that the frequency function describing the universe sampled is linear for each class interval, the parameters of the line corresponding to a given interval can readily be estimated from the number of observations and the sum of the measurements for the interval. The author shows how to estimate the standard deviation and higher moments and evaluates the error made.

W. Feller.

Pierce, Joseph A. Correction formulas for moments of a grouped-distribution of discrete variates. J. Amer. Statist. Assoc. 38, 57–62 (1943). [MF 8137]

This paper is based on ideas due to H. L. Jones [cf. the paper reviewed above] and P. S. Dwyer [Ann. Math. Statistics 13, 138–155 (1942); these Rev. 4, 24] and essentially adds nothing to them, though it does give in detail the means of calculating the tth moment of a grouped distribution without error by means of auxiliary information recorded at the time the grouped distribution is formed.

C. C. Craig (Ann Arbor, Mich.).

Kolmogoroff, A. Sur l'estimation statistique des paramètres de la loi de Gauss. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 6, 3–32 (1942). (Russian. French summary) [MF 7694]

Primarily this is an article on the theory of confidence intervals, written as an answer to inquiries of artillery men as to the most advisable means of estimating the parameters involved in the normal probability law. The author begins by considering the conditions under which the classic approach, by means of Bayes' formula, is applicable. In so doing he gives a new definition of sufficient statistics and of sufficient sets of statistics, not equivalent to the usual one. Let $x_1, \cdots, x_n$ be the observable random variables. A set of statistics $T_1, T_2, \cdots, T_m$ is called sufficient with respect to the system of parameters $\theta_1, \theta_2, \cdots, \theta_k$ if, whatever be the a priori probability law of these parameters, their a posteriori probability law, given all the observable random variables $x_1, \cdots, x_n$, depends on $T_1, T_2, \cdots, T_m$ but not explicitly on $x_1, x_2, \cdots, x_n$. The following example illustrates the difference between the Kolmogoroff definition of sufficiency and the usual one. Assume that all the x's follow the same normal law with an unknown mean ξ and an unknown standard error σ. According to the usual definition, $\bar{x}=\sum x_i/n$ is a specific sufficient statistic for ξ. This is not the case if sufficiency is taken in the sense of Kolmogoroff.

In connection with Bayes' approach to the problem of estimation the author gives three theorems concerning the limiting form of the a posteriori distribution (1) of the mean ξ of a normal population, (2) of the precision factor h of a normal population and (3) of both these parameters. All three theorems give estimates of the order of the remainder term, based on the assumption that the a priori probability law of the parameters considered is not only continuous [as in the known theorem of von Mises] but has bounded derivatives of first order with respect to the parameters concerned. The inequalities for the remainder terms are sharper than those deducible only on the assumption of continuity. It is shown also that they cannot be made any more sharp by additional hypotheses concerning the derivatives of higher orders.

Having found that the approach to the problem of estimation by means of Bayes' theorem is of a limited applicability, the author proceeds to state the problem of confidence intervals and to clarify the misconceptions concerning this theory that appear in the literature from time to time. A systematic search of confidence limits in general conditions is postponed to another article. In the paper under review the author again finds the familiar confidence intervals for the mean and the standard error of a normal population, known to be the short unbiased. To simplify the solution the author begins with the following "natural" principles: (i) that the confidence limits should depend only on sufficient systems of statistics (in the new sense) and (ii) that they be invariant under a linear transformation of parameters. The last pages of the paper deal with the problem of single estimates, particularly connected with problems of artillery.

J. Neyman (Berkeley, Calif.).

Aroian, Leo A. A new approximation to the levels of significance of the chi-square distribution. Ann. Math. Statistics 14, 93–95 (1943). [MF 8254]

The author determines percentage points for the χ^2-distribution by determining the corresponding standard deviate for χ^2 as a quadratic function of the skewness (α_3). The coefficients are fitted by least squares using 0, ±0.1, ±0.2, ±0.3 and ±0.4 as values of α_3 and the constant is then adjusted to give the correct value for $\alpha_3=0$. A numerical comparison of results for a number of degrees of freedom of 30 or more is quite favorable to this method in this range and it is pointed out that no interpolation is required in its application.

C. C. Craig (Ann Arbor, Mich.).

Scheffé, Henry. On solutions of the Behrens-Fisher problem, based on the t-distribution. Ann. Math. Statistics 14, 35–44 (1943). [MF 8277]

Given two samples $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_n)$ $(m\leqq n)$ from two normal populations, the ratio of the variances of which is unknown, the problem here dealt with is the interval estimation of δ, the difference between the population means, by means of the t-distribution. The first attack is to use, in forming t, the variables $d_i=x_i-\sum_{j=1}^{n}c_{ij}y_j$, $i=1, 2, \cdots, m$, the c_{ij}'s being chosen so that

$E(d_i)=\delta$, $\sigma_{d_i}^2=\sigma^2$ and $E(d_id_j)=0$, $i\neq j$, for $i=1, 2, \cdots, m$. Subject to these conditions, it is shown further how to choose the c_{ij}'s specifically in order to minimize the confidence interval determined by t, which here has $m-1$ degrees of freedom. In an unpublished solution of this problem by M. S. Bartlett referred to by B. L. Welch [Biometrika **29**, 360–362 (1938)], c_{ij} was set equal to δ_{ij} (the Kronecker δ); in that case the expected length of the confidence interval is found to be greater than that given here if $m\neq n$. In a second section of the paper the problem is approached in a more general fashion; the variables used in setting up the t are differences between linear forms in the x's and the y's. By an interesting argument it is proven that it is impossible to obtain a t with more than $m-1$ degrees of freedom, and it readily follows that the solution first found is the best possible. Finally, a numerical comparison of the confidence intervals here found with the optimum ones obtainable in case the ratio of the population variances is known shows that for $m>10$ the solution given here is quite efficient. *C. C. Craig* (Ann Arbor, Mich.).

Wallis, W. Allen. Compounding probabilities from independent significance tests. Econometrica **10**, 229–248 (1942). [MF 7182]

Denote by $x_1, x_2, \cdots, x_N$ statistical test criteria used to test the same hypothesis on N different and independent series of observations. Let x_i' be the observed value of x_i and p_i the probability of $x_i>x_i'$. The paper is concerned with Fisher's method of combining all the separate tests into a single one, consisting in the rule of rejecting the hypothesis tested when the product $Q=\prod_{i=1}^N p_i$ is sufficiently small, namely, when $-2\log Q>\chi_\epsilon^2$, where χ_ϵ^2 is the tabled value of χ^2 with $2N$ degrees of freedom, corresponding to the level of significance ϵ. The first part of the paper deals with the case where the distributions of all the x_i's are continuous. The author describes some misconceptions connected with the method. It is also shown that, with a plausible set of hypotheses alternative to the one tested, the described test is likely to possess a satisfactory power. Essentially new results are given in the second part of the paper, relating to the case where one or more of the criteria x_i is discontinuous. Considering again the product Q as the combined criterion, the author deduces formulae giving its distribution. A few examples show that in this case a direct application of the method, involving a reference to the tables of χ^2 distribution, is likely to seriously underestimate significance. *J. Neyman* (Berkeley, Calif.).

Peiser, Alfred M. Asymptotic formulas for significance levels of certain distributions. Ann. Math. Statistics **14**, 56–62 (1943). [MF 8249]

Asymptotic formulae that give close approximations for samples as small as ten are given for the p per cent points of the chi-square and Student's t distributions with n degrees of freedom in terms of the p per cent point of the normal distribution and the number of degrees of freedom. *W. G. Madow* (Washington, D. C.).

Nair, U. Sivaraman. A comparison of tests for the significance of the difference between two variances. Sankhyā **5**, 157–164 (1941). [MF 7351]

Consider two normal populations with unknown means and unknown standard deviations σ_1 and $\sigma_2=\sigma_1/\theta$. Let S_i denote the sum of squares of deviations

$$\sum_{i=1}^{n_i}(x_{t_i}-\bar{x}_i)^2$$

for the sample from the ith population. Finally, let $u=S_1/(S_1+S_2)$. The author compares the power functions of five alternative tests of the hypothesis H_0 that $\theta=1$. These tests are: (i) reject H_0 when $u<U_1=\text{const.}$; (ii) reject H_0 when $u>U_2=\text{const.}$ (these are the known uniformly most powerful tests of H_0 with respect to partial sets of admissible hypotheses, either of $\theta\geqq 1$ or $\theta\leqq 1$); (iii) reject H_0 when either $u<u_1'$ or $u>u_2'$, where the u_k' are so selected that

$$P\{u<u_1'|H_0\}=P\{u>u_2'|H_0\}=\alpha/2,$$

with α denoting the level of significance (this is the so-called equal tail area test frequently applied to test H_0 against the general set of admissible hypotheses ascribing to θ any positive value); (iv) reject H_0 when either $u<u_1''$ or $u>u_2''$, where u_1'' and u_2'' are so selected as to make the test unbiased, the condition of unbiasedness being

$$u_1''^{(n_1-1)}(1-u_1'')^{(n_2-1)}=u_2''^{(n_1-1)}(1-u_2'')^{(n_2-1)};$$

(v) the likelihood ratio test of H_0, which is of the same general form as (iii) and (iv) except that the critical values of u, u_1''' and u_2''' satisfy the condition

$$u_1'''^{n_1}(1-u_1''')^{n_2}=u_2'''^{n_1}(1-u_2''')^{n_2}.$$

All the tests, except (iv), appear to be biased with test (iii) being more satisfactory than the others. *J. Neyman.*

Wald, Abraham. An extension of Wilks' method for setting tolerance limits. Ann. Math. Statistics **14**, 45–55 (1943). [MF 8278]

This paper provides a method for setting tolerance limits for two or more statistically dependent variables wherein the procedure of choosing the tolerance limits is independent of the unknown probability density function of the variates. The bivariate case is treated in some detail and helpful numerical illustrations are included. *W. A. Shewhart.*

Lawley, D. N. On problems connected with item selection and test construction. Proc. Roy. Soc. Edinburgh. Sect. A. **61**, 273–287 (1943). [MF 8302]

It is assumed that a test is composed of items (on which the subject scores 0 or 1) all of which measure the same ability, and that this ability is normally distributed in the subjects who are tested. From the results from a group of subjects of known abilities, estimates of the mean and variance for any one item of the test are obtained by an application of maximum likelihood, the approximate solution of the resulting equations leading to the previously used "constant process." If it further is true that the test contains a rather large number of items with their mean scores normally distributed and their variances constant, further results follow, using suitable approximation methods. In particular, normally the true abilities of subjects are not known but a reciprocity is pointed out between the number of items answered correctly by a subject and the number of subjects answering a given item correctly. This relation then is used to obtain estimates of the mean of item means, the variance of these means and the common variance of scores on each item, from which the reliability of the test can be estimated. There is a numerical illustration of this last method. *C. C. Craig* (Ann Arbor, Mich.).

Bose, R. C. and Kishen, K. On the problem of confounding in the general symmetrical factorial design. Sankhyā **5**, 21–36 (1940). [MF 7346]

A procedure based on a generalization of Barnard's definition of the generalized interaction is given for forming

subblocks of s^{m-k} plots ($k=1, 2, \cdots, m-1$) in the case of factorial designs of type s^m. The procedure is applied in enumerating all the different types of confounding occurring when factorial designs of type s^3 and s^4 are arranged in blocks containing s and s^2 plots. *W. G. Madow.*

Nair, K. R. Balanced confounded arrangements for the 5^n type of experiment. Sankhyā 5, 57–70 (1940). [MF 7347]

The author gives an enumeration of confounded arrangements in experiments of type 5^n, which is a special case of the general one of type p^n, p prime or a power of a prime. The method was developed for the general case by the author in an earlier paper [Sankhyā 4 (1938)] and depends on interchanges obtainable by Hyper-Graeco-Latin squares. The enumeration of the confounded arrangements for experiments of type 3^n and 4^n was also given in the earlier paper. *W. G. Madow* (Washington, D. C.).

Gage, Robert. Contents of Tippett's "Random Sampling Numbers". J. Amer. Statist. Assoc. 38, 223–227 (1943). [MF 8437]

Swed, Frieda S. and Eisenhart, C. Tables for testing randomness of grouping in a sequence of alternatives. Ann. Math. Statistics 14, 66–87 (1943). [MF 8251]

Feather, N. On the statistics of random distributions of paired events, with applications to the results obtained in the use of the interval selector with particle counters. Proc. Cambridge Philos. Soc. 39, 84–99 (1943). [MF 8111]

Consider the time interval t between one response of a particle counter and the next; let the expectation of t be $1/N$, and let $W(s)=\text{Prob}\,(t<s)$. Then, under the usual hypotheses, (1) $W(t)=1-e^{-Nt}$. The author examines various alternative hypotheses, more appropriate physically, and discusses the general behaviour of $W(t)$ in the light of the resulting formulae. Suppose, first, that "primary" events occur according to a law of the type (1) with n in place of N, that each is followed by a "secondary" after an interval τ and that the counter responds to both kinds. Let $\Phi(s)=\text{Prob}\,(\tau<s)$. Then $N=2n$ and

$$(2)\qquad 2W(t)=2-[2-\Phi(t)]\exp\left[-2nt+n\int_0^t\Phi(\tau)d\tau\right].$$

In particular, we may assume, for example, (i) $\tau=$constant or (ii) $\Phi(\tau)=1-e^{-\lambda\tau}$. Other hypotheses considered limit the occurrence of secondaries, and some provide for "tertiaries," etc. The author also investigates briefly the loss due to the "resolving-time" of the counter. [The paper contains a few slips and misprints, for example, in formulae (xi) and (xiv').] *H. P. Mulholland* (Newcastle, England).

TOPOLOGY

Albuquerque, J. La notion de "bord" en topologie. Portugaliae Math. 3, 185–200 (1942). [MF 7910]

In a topological space (in the sense of Fréchet) the primitive notion is closure, the bord of a set X is defined by $b(X)=X\cdot\overline{1-X}$, the orle by $0(X)=\bar{X}(1-X)$ and a space (F) is one characterized by the axioms $\bar{0}=0$, $X\subset\bar{X}$, $\bar{X}+\bar{Y}\subset\overline{X+Y}$, $\bar{\bar{X}}=\bar{X}$. In this paper the author takes as primitive notion first bord and then orle and for each characterizes not only the spaces (F) but also the quasi-ordered spaces of G. Birkhoff, the discrete spaces of Alexandroff and the Hausdorff spaces known as regular, as normal and as completely normal. *J. F. Randolph.*

Monteiro, António et Ribeiro, Hugo. L'opération de fermeture et ses invariants dans les systèmes partiellement ordonnés. Portugaliae Math. 3, 171–184 (1942). [MF 7909]

Let P be a partially ordered system (in the sense of G. Birkhoff) and let φ be a function on P to P. An element I of P is said to be an invariant of φ if $\varphi(I)=I$. The function φ is said to be a closure operation if axiom I is satisfied. Axiom I: $X\varepsilon P$ implies $X\subset P(X)$. The fundamental theorem of the paper is: in order that a closure operation φ defined on P be uniquely determined by its family of invariants it is necessary and sufficient that φ satisfy axioms II and III. Axiom II: $X\varepsilon P$, $Y\varepsilon P$, $X\subset Y$ imply $\varphi(X)\subset\varphi(Y)$. Axiom III: $X\varepsilon P$ implies $\varphi[\varphi(X)]=\varphi[X]$. The couple $[P,\varphi]$, where φ satisfies axioms I, II and III, is said to be a topological partially ordered system. The properties of the invariants of such a system are studied and then special cases are considered according as P has a last element or each subfamily of P has a greatest lower bound. Let Φ be the family of topologies of P; that is, Φ is the set each of whose elements is in the form $[P,\alpha]$, where α satisfies axioms I, II and III. Thus $[P,\theta]\varepsilon\Phi$, where θ is defined by $\theta(X)=X$ whenever $X\varepsilon P$. An element $[P,\alpha]$ of Φ is said to precede $[P,\beta]\varepsilon\Phi$, $[P,\alpha]\leqq[P,\beta]$, if and only if $\alpha(X)\leqq\beta(X)$ for each $X\varepsilon P$. Under this ordering the set Φ is shown to be a partially ordered system with $[P,\theta]$ as its "first" element. Further properties of P that carry over to Φ are given such as: if P is a complete structure, then Φ is a complete structure.
J. F. Randolph (Ithaca, N. Y.).

Eilenberg, Samuel. Banach space methods in topology. Ann. of Math. (2) 43, 568–579 (1942). [MF 7011]

This paper concerns the relation between the topology of a space X and the metric of the Banach space $B(X)$ whose elements are the bounded real continuous functions on X. The principal result is that, if X and Y are completely regular and satisfy the first denumerability axiom, then $B(X)$ and $B(Y)$ are isometric if and only if X and Y are homeomorphic. This extends a similar theorem of Banach [Théorie des opérations linéaires, Warsaw, 1932, p. 170] in which X is assumed to be metric compact, and its generalization by Stone [Trans. Amer. Math. Soc. 41, 375–481 (1937); in particular, p. 469] for compact X is also obtained. The method develops a space Ξ, homeomorphic to X, whose elements are defined in terms of maximal convex subsets of the surface of the unit sphere in $B(X)$. The result is applied to show that, if the number of components of X is finite, $B(X)$ may be decomposed into the same number of direct summands, and conversely. *L. W. Cohen.*

Blumenthal, Leonard M. New characterizations of segments and arcs. Proc. Nat. Acad. Sci. U. S. A. 29, 107–109 (1943). [MF 8211]

As a contribution toward the solution of the general problem of determining the metric and topological properties of various classes of metric spaces when these spaces are assumed to be free from equilateral k-tuples and the topo-

logical properties a space must possess in order that it may be homeomorphic to a space with prescribed metric properties, the author establishes the following two theorems. Theorem 1. A compact convex metric space M of at least two points is congruent with a line segment if and only if M is free from equilateral triples. Theorem 2. A metric Peano continuum without an equilateral triple is an arc. It had been shown previously by Blumenthal and Robinson [Rep. Math. Colloquium (2), no. **2**, 3 pp. (1940); these Rev. **1**, 263] that a complete, convex, externally convex metric space M (containing at least two points) is metrically a straight line if and only if M does not possess an equilateral triple. *G. B. Price* (Lawrence, Kan.).

Radó, Tibor. On continuous path-surfaces of zero area. Ann. of Math. (2) **44**, 173–191 (1943). [MF 8282]

Let $\mathfrak{z}=\mathfrak{z}(P)$, P on the surface U of a unit sphere, define a path surface S. Let Σ^* denote the upper-semicontinuous collection of maximal continua on each of which $\mathfrak{z}(P)$ is constant. This is a Peano space if topologized in the obvious way and is, as R. L. Moore has shown, a cactoid. The writer proves that a necessary and sufficient condition that the Lebesgue area of S be zero is that Σ^* be a dendrite. As the writer remarks, the result is deducible from previous results of the reviewer. However, the proof here given is simple and elegant and involves only the definition of area and well-known concepts and theorems in topology.

C. B. Morrey, Jr. (Aberdeen, Md.).

Kiang, Tsai-han. Remarks on two-leaved orientable covering manifolds of closed manifolds. Ann. of Math. (2) **44**, 128–130 (1943). [MF 8080]

Proofs of the two theorems: (1) an orientable covering manifold of a nonorientable manifold M is a covering manifold of the 2-sheeted orientable covering manifold of M, and (2) for an orientable manifold $\bar{M}$ the existence of an orientable (nonorientable) manifold M, for which $\bar{M}$ is a 2-sheeted covering manifold, is equivalent to the existence of a homeomorphism of $\bar{M}$ with itself, which is involutory, without fixed points and orientation-preserving (orientation-reversing). *H. Samelson* (Laramie, Wyo.).

James, R. C. Linearly arc-wise connected topological Abelian groups. Ann. of Math. (2) **44**, 93–102 (1943). [MF 8078]

A topological Abelian group T is called linearly arcwise connected if for any x in T there exists a continuous function $f(t)$, $0\leqq t\leqq 1$, such that $f(t_1+t_2)=f(t_1)+f(t_2)$, $f(1)=x$, $f(0)=0$; T is uniquely arcwise connected if the linear path $f(t)$ is unique. In a similar manner the concept of local linear arcwise connectedness is defined. The author proves that, if T is connected, simply connected and locally linearly arcwise connected, then either T has no elements of finite order or for each $n>1$ the set of elements of order n is dense in itself. A number of conditions that T be a linear topological space are given. We note the following. A necessary and sufficient condition that a topological Abelian group be a linear topological space is that it be connected, locally linearly arcwise connected and possess no elements of finite order. Definitions of convexity, boundedness of subsets and normability of Abelian topological groups are given and conditions that T be a convex linear topological space and that T be a normable linear topological space are derived.

N. Jacobson (Chapel Hill, N. C.).

Whyburn, G. T. On the interiority of real functions. Bull. Amer. Math. Soc. **48**, 942–945 (1942). [MF 7517]

The first part of the paper deals with the problem of ascertaining the amount of interiority of a continuous mapping. The basic theorem is that, if $f(x)$ is a real-valued continuous function defined on a locally connected separable metric space M (therefore transforming M into a set Y of real numbers), there exists a countable subset C of Y such that f is interior at every point of $M-f^{-1}(C)$. Extensions to more general cases are obtained; for instance, if K is a graph and f is a continuous mapping of M onto K, then f is interior at all points of $M-f^{-1}(C)$, where C is some countable subset of K. In the second part of the paper the author suggests the problem of extending a continuous mapping so as to carry over the interiority property to the extension. Results are obtained in cases where M is a subcontinuum of a locally connected continuum having certain cyclic properties. *R. L. Wilder* (Ann Arbor, Mich.).

Fox, R. H. On homotopy type and deformation retracts. Ann. of Math. (2) **44**, 40–50 (1943). [MF 8073]

It is noted that the concept of homotopy type splits naturally into the two concepts called right- and left-homotopy inversion, which are shown to correspond, respectively, to deformation and retraction. Using the mapping cylinder as a fundamental tool, it is shown that, if X, Y, Z are topological spaces, θ a mapping of X into Z and f a mapping of X into Y, then there is a mapping g of Y into Z satisfying $gf\cong\theta$ if and only if θ can be extended to $Y+C_f$. Also, under similar definitions of X, Y, Z, θ, if g is a mapping of Y into Z, then there is a mapping f of X into Y satisfying $gf\cong\theta$ if and only if θ is homotopic in $Z+C_g$ to a map of X into Y. These theorems are then applied to the Hopf-Pannwitz deformation, and certain specializations are considered which tend to bridge the gaps between homotopy type and nucleus and between homotopy type and topological type. Finally n-dimensional analogues of the above theorems are considered. *H. M. Gehman.*

Fox, Ralph H. On the deformation retraction of some function spaces associated with the relative homotopy groups. Ann. of Math. (2) **44**, 51–56 (1943). [MF 8074]

A study is made of the relationship between deformation retraction in a function space and deformation retraction of an image space. A theorem of Hurewicz is generalized to give the conditions under which one function is deformable into another. Three lemmas are given which are contributions to the study of fibre spaces. *H. M. Gehman.*

Eilenberg, Samuel and MacLane, Saunders. Relations between homology and homotopy groups. Proc. Nat. Acad. Sci. U. S. A. **29**, 155–158 (1943). [MF 8334]

The relations in question (stated without proof) are generalizations of results recently obtained by H. Hopf [Comment. Math. Helv. **14**, 257–309 (1942); **15**, 27–32 (1942); these Rev. **3**, 316; **4**, 173]. The methods used here give explicit descriptions of all the groups involved. Let P be a connected locally finite simplicial complex, $\mathfrak{H}^q(P, G)$ the homology group of finite q-chains of P over a discrete additive group G, $H_q(P, G)$ the cohomology group of infinite

q-cochains of P over a topological additive group G. Let $\pi_q(P)$ be the qth homotopy group of P, $\mathfrak{S}^q(P, G)$ the subgroup of $\mathfrak{H}^q(P, G)$ determined by the cycles of P that can be obtained from cycles on the q-sphere S^q by continuous mappings of S^q into P. Given a discrete group π written multiplicatively, let $K(\pi)$ be the abstract complex defined as follows: the q-cells of $K(\pi)$ are the $(q+1)$-rowed matrices $\Delta = \|p_{ij}\|$, where the p_{ij} are elements of π such that $p_{ij}p_{jk} = p_{ik}$; the boundary operator α is given by $\alpha\Delta = \Sigma(-1)^i\Delta^{(i)}$, where $\Delta^{(i)}$ is the matrix obtained from Δ by striking out the ith row and column. $K(\pi_i(P))$

Main theorem: if $\pi_i(P) = 0$ for $1 < i < n$, then

$$\begin{aligned} \mathfrak{H}^q(P, G) &\cong \mathfrak{H}^q(\pi_1(P), G) \qquad \text{for } q < n, \\ H_q(P, G) &\cong H_q(\pi_1(P), G) \qquad \text{for } q < n, \\ \mathfrak{H}^n(P, G)/\mathfrak{S}^n(P, G) &\cong \mathfrak{H}^n(\pi_1(P), G). \end{aligned}$$

The authors outline a method for computing the groups $\mathfrak{H}^q(\pi, G)$ and $H_q(\pi, G)$, leading to the particular results that $H_1(\pi, G)$ is the group of homomorphisms $\pi \to G$ and $H_2(\pi, G)$ is the group of central extensions of G by π. The main theorem admits a generalization in which it is assumed that $\pi_i(P) = 0$ for $1 < i < m$ and for $m < i < n$. The complex $K(\pi)$ must now be replaced by a certain complex $K(\pi, m)$ and $\pi_1(P)$ by $\pi_m(P)$. *P. A. Smith* (New York, N. Y.).

Mayer, W. and Campbell, A. D. Generalized homology groups. Univ. Nac. Tucumán. Revista A. 3, 155–194 (1942). [MF 8151]

The paper develops fully a previous brief note of the authors [Proc. Nat. Acad. Sci. U. S. A. 26, 655–656 (1940); these Rev. 2, 75]. The groups studied are the same as those of W. Mayer [Ann. of Math. (2) 43, 370–380, 594–605 (1942); these Rev. 3, 318] except that only the case $p=3$ is considered, that is, the case when the boundary operator F satisfies $FFF=0$. The homology groups are defined for a simplicial complex and also for a topological space using singular simplexes. Then, using barycentric subdivisions and standard deformation devices, the invariance theorem is established for the new groups. *S. Eilenberg.*

Begle, Edward G. Intersections of contractible polyhedra. Bull. Amer. Math. Soc. 49, 386–387 (1943). [MF 8383]

An example is given of polyhedra A, B, C, D with the following properties: (1) $C = A \cup B$, (2) $D = A \cap B$, (3) A, B, and C are contractible, and (4) D is not contractible. Using this example one can construct compacta A', B', C', D' such that (1) $C' = A' \cup B'$, (2) $D' = A' \cap B'$, (3) A', B', and C' are locally contractible, and (4) D' is not locally contractible. *R. H. Fox* (Urbana, Ill.).

MECHANICS

García, Godofredo. On the gyroscopic effect in the motion of a projectile. Actas Acad. Ci. Lima 5, 79–86 (1942). (Spanish) [MF 8029]

Neglecting the forces due to lateral friction and to changes of orientation of the axis, the author presents an approximate formula for the velocity of precession of a projectile

$$\psi' = \frac{flR}{[(C-A)\varphi' + C\Omega_0]\cos\varphi}.$$

In this formula R denotes the force of resistance, l the distance along the axis from the center of resistance to the center of gravity, f the ratio of the sines of the inclinations of R and of the tangent along the path to the axis, φ the angle of "proper motion," Ω_0 the angular velocity about the axis and C and A the equatorial and axial moments of inertia. If R is parallel to the tangent $f=1$ and if φ is small and φ' negligible, the formula reduces to $\psi' = lR/C\Omega_0$.
W. E. Milne (Corvallis, Ore.).

Kimball, W. S. The path equation for motion on the surface of the rotating earth in a uniform parallel field of force with initial velocity along the field. J. Franklin Inst. 235, 273–283 (1943). [MF 8140]

Solution of a problem of motion, assuming the author's special additional Coriolis' term K [same J. 234, 453–472 (1942); these Rev. 4, 116]. The reviewer finds the discussion of this term K unsatisfactory and doubts if it has any physical significance. *P. Franklin.*

Četajev, N. G. On the equations of Poincaré. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] 5, 253–262 (1941). (Russian. English summary) [MF 7709]

The paper presents an extension of the equations of Poincaré to the case of an intransitive group of possible displacements. Using the symbols of Cartan, an intransitive group of possible displacements is determined for linear holonomic geometrical constraints. The equations of motion are given in the form of Poincaré, in canonic form and in the form of equations in partial derivatives of the first order. Furthermore, the author investigates general properties of cyclic displacements and properties of Hamilton's function of action. *W. Hurewicz* (Providence, R. I.).

Malkin, I. G. Basic theorems of the theory of stability of motion. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 6, 411–448 (1942). (Russian. English summary) [MF 8361]

The author examines problems of s. (stability), a.s. (asymptotic stability) and i. (instability) in the sense of L. (Liapounoff) for equations of the form (1) $x_s' = \varphi_s(t, x_1, \cdots, x_m)$ $(s=1, \cdots, m)$, prime denoting differentiation with respect to t; the φ_s are power series in $x_1, \cdots, x_m$, with bounded continuous coefficients possibly dependent on t, convergent for $t \geqq t_0$, $|x_s| \leqq h$; $\varphi_s = 0$ for $x_1 = \cdots = x_m = 0$. If the coefficients of the linear part of φ_s are constants p_{sk}, the nature of the roots ρ_j of the determinantal equation $|(p_{sk} - \delta_{sk}\rho)| = 0$ $(\delta_{ss}=1;\ \delta_{sk}=0$ for $s \neq k)$ is of importance. L. has shown that the unperturbed motion is a.s. if the $\Re\rho_j$ (real parts of the ρ_j) are all less than 0; if at least one $\Re\rho_j > 0$, we have i. Serious difficulties arise in the singular cases when all the $\Re\rho_j \leqq 0$, while some $\Re\rho_j = 0$. L. treated some very special cases of the latter. Subsequent literature, basing itself mostly on L.'s methods, pushed studies of this type somewhat further, leaving, however, important gaps. The author makes the best advance up to date in the singular theory, succeeding substantially on the basis of a new method. The following is a typical result. Let

$$\text{(2)} \qquad \begin{aligned} y_i' &= r_{i1}x_1 + \cdots + r_{in}x_n + Y_i(t, x, y) = \bar{Y}_i, \quad i=1, \cdots, k, \\ x_s' &= p_{s1}x_1 + \cdots + p_{sn}x_n + X_s(t, x, y) = \bar{X}_s, \quad s=1, \cdots, n, \end{aligned}$$

be a system for perturbed motion, the linear parts of $\bar{Y}_i$, $\bar{X}_s$ being displayed and the r_{ij}, p_{sj} being bounded functions of t; suppose there exists a quadratic form $V(t, x)$ in $x_1, \cdots, x_n$, with coefficients depending continuously and boundedly on

t, so that $V \geqq a^2r^2$ $(r^2 = x_1^2 + \cdots + x_n^2)$,

$$V_t + \sum V_{x_s}(p_{s1}x_1 + \cdots + p_{sn}x_n) \leqq -b^2r^2.$$

Suppose the solution $y_1 = \cdots = y_k = 0$ of $y_i' = Y_i(t, y_1, \cdots, y_k, 0, \cdots, 0)$ is s. or a.s. or i. for every choice of terms of order greater than N. Then, if the $X_s(t, y_1, \cdots, y_k, 0, \cdots, 0)$ begin with terms of order not less than N, the solution $x_1 = \cdots = x_n = y_1 = \cdots = y_k = 0$ of (2) is correspondingly s., a.s. or i. The author gives corresponding results for the case of periodic φ_s in (1). *W. J. Trjitzinsky.*

Hartman, Philip and Wintner, Aurel. Integrability in the large and dynamical stability. Amer. J. Math. **65**, 273–278 (1943). [MF 8203]

In this paper the authors demonstrate that the features of the dynamical systems on the surface of a torus [Poincaré, Oeuvres Completes, vol. I, Paris, 1928, pp. 137–158; Kneser, Math. Ann. **91**, 135–154 (1924); Denjoy, J. Math. Pures Appl. (9) **11**, 333–375 (1932)] involve a fundamental relationship between stability and almost periodicity which can be generalized to a broad class of dynamical systems. A solution curve $x = x(t, x_0)$ of a general dynamical system is here termed stable if there exists a positive function $\delta(x_0, \epsilon)$, $\epsilon > 0$, such that, for every solution $y = y(t, y_0)$ which, for some value of t, satisfies the condition

$$|y(t, y_0) - x(t, x_0)| < \delta(x_0, \epsilon),$$

one has

$$|y(t, x_0) - x(t, x_0)| < \epsilon$$

for all t. [This definition is highly restrictive; for example, $x = x(t, x_0)$ fails to be stable if there is a $y(t, y_0)$ such that

$$\liminf_{t\to\infty} |y(t, y_0) - x(t, x_0)| = 0. \quad]$$

The following theorem is then proved: if a dynamical system possesses a compact invariant set S, and if there is a solution of the system on S and regionally transitive (=everywhere dense) on S, then that solution is stable on S if and only if it is almost periodic; in such case every solution on S is stable and almost periodic, the stability function δ can be chosen independent of x_0 on S and all solutions on S have the same translation classes for each ϵ (so that the Fourier series $\sum a_m e^{i\Lambda_m t}$ of different solutions differ only in the values of the quantities arg a_m); furthermore a Lebesgue measure can then be defined for Borel sets on S in terms of which the given flow is metrically transitive. *W. Kaplan* (Ann Arbor, Mich.).

Kasner, Edward and DeCicco, John. The geometry of velocity systems. Bull. Amer. Math. Soc. **49**, 236–245 (1943). [MF 8218]

This paper presents some geometric and variational properties of velocity systems. As appropriate to systems related to the conformal group, minimal coördinates $u = x + iy$, $v = x - iy$ are used. In terms of these coördinates, the differential equation of a velocity system takes the form $v'' = c(u, v)v' - d(u, v)v'^2$. *P. Franklin.*

Relativity, Astronomy

Coxeter, H. S. M. A geometrical background for de Sitter's world. Amer. Math. Monthly **50**, 217–228 (1943). [MF 8212]

The author presents an introduction to Minkowski's space-time geometry of 1+2 dimensions, regarded as the geometry in an affine space with a parabolic metric imposed by a hyperbolic polarity in the plane at infinity. Then he proceeds to the discussion of de Sitter's world, which he describes as the space exterior to an oval quadric threefold in projective four-space with the metric determined by that quadric threefold. He explains Eddington's mapping of a two-dimensional de Sitter world on a one sheeted Minkowski-sphere in which two antipodal points are identified. A few properties of de Sitter's world which can be derived from its projective model are examined in the last sections of the paper. *E. Helly* (Chicago, Ill.).

Einstein, A. and Pauli, W. On the non-existence of regular stationary solutions of relativistic field equations. Ann. of Math. (2) **44**, 131–137 (1943). [MF 8279]

The authors generalize a result previously obtained by A. Einstein [Univ. Nac. Tucumán. Revista A. **2**, 5–15 (1941); these Rev. **4**, 55] for a four dimensional space. In this paper they consider an n dimensional metric $ds^2 = g_{ik}dx^i dx^k$ which is independent of the coordinates x^ρ $(\rho = 4, 5, \cdots, n)$ and is such that the metric of the three dimensional sub-space is positive definite. For nonsingular solutions of the field equations, it is shown that the deviations of the g_{ik} from their Euclidean values are at least of order $1/r^2$, where $r^2 = \sum_1^3 (x^i)^2$. From this result, the authors conclude that, in Kaluza's five dimensional theory, there cannot exist a nonsingular solution which represents the field of a nonvanishing total mass or charge. *M. Wyman* (Edmonton, Alta.).

Lichnerowicz, André. Sur une généralisation des espaces de Finsler. C. R. Acad. Sci. Paris **214**, 599–601 (1942). [MF 7892]

This note contains an extension of the results published previously in the same C. R. **212**, 328–331 (1941); cf. these Rev. **3**, 63. *T. Y. Thomas* (Los Angeles, Calif.).

Dive, Pierre. Propagation ellipsoïdale des ondes électromagnétiques. C. R. Acad. Sci. Paris **214**, 612–615 (1942). [MF 7898]

The Einstein contraction can be obtained by assuming that c depends on the motion of the source, but is symmetric with respect to it. [This is an idea originally due to Poincaré.] The writer develops its consequences for electrodynamics. *D. G. Bourgin* (Urbana, Ill.).

Milne, E. A. Rational electrodynamics. I. The limitations of classical electromagnetism. Philos. Mag. (7) **34**, 73–82 (1943). [MF 8130]

This paper, the first of a series of papers, has an introductory character. Important results are only announced, their deduction to be given later. They concern a new electrodynamics based on the author's "kinematical relativity." In more abbreviated form, these results have already appeared in two papers [Proc. Roy. Soc. London. Ser. A. **165**, 313–332, 333–357 (1938)]. The scope of problems and importance of their solutions is best characterized in the author's own words: "In particular, it will be shown how the point-singularity that is an electron behaves as though possessing a definite radius, the classical 'radius of the electron,' in spite of the fact that in the new theory the inverse square law of Coulomb holds down to indefinitely small distances. It will be shown further how an analysis of the possible orbits of two charges of opposite sign admits the existence of a neutron, as well as the known Bohr orbits; this will lead to some considerations on intranuclear dy-

namics. The progress that will be made is intimately bound up with the construction of an electrodynamics in which the fundamental laws and equations are not mere inductions from macroscopic laws but should be valid at particle-velocities indefinitely close to that of light; the electrodynamics is linked at each stage with high-velocity rational dynamics. . . . Lastly, it will be shown that the new theory removes some of the difficulties that have gathered around the theory of Faraday and Clark Maxwell."

L. Infeld (Toronto, Ont.).

Milne, E. A. Rational electrodynamics. II. The ideas of kinematical relativity. Philos. Mag. (7) **34**, 82–101 (1943). [MF 8131]

In this paper the author reviews once more the ideas of kinematical relativity. They were presented before more fully in a book and many papers by the author which he quotes at the end of the first introductory article [see the paper reviewed above]. Especially stressed is the deduction of the gravitational law from the author's theory. In Milne's own words: "The expression of the inverse square law, or its potential $1/r$, in a form relativistically invariant under Lorentz transformation . . . is of course revolutionary."

L. Infeld (Toronto, Ont.).

Buchanan, H. E. The present state of the three body problem. Revista Mat. Hisp.-Amer. (4) **2**, 97–103, 211–217, 247–252 (1942). (Spanish) [MF 7158]

Hinrichsen, J. J. L. The libration points in an n-body problem. Amer. Math. Monthly **50**, 231–237 (1943). [MF 8214]

This paper considers the location of the libration points for the restricted n-body problem such that one mass is infinitesimal while the other $n-1$ masses are equal and form the vertices of a regular polygon rotating with constant angular velocity; it thus continues research in a field in which there is a considerable literature [see, in particular, O. Dziobek, Astr. Nachr. **152**, 34–46 (1900); further references are given in E. T. Whittaker, Analytical Dynamics, 4th ed., Cambridge University Press, 1937, p. 409 and in A. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton Mathematical Series, vol. 5, Princeton University Press, Princeton, N. J., 1941, p. 430; these Rev. **3**, 215]. Detailed estimates are given for $n=4$ and $n=5$ and generalizations for arbitrary n are stated without proof. The case $n=4$ was analyzed fully by F. R. Moulton [Trans. Amer. Math. Soc. **1**, 17–29 (1900)]. *W. Kaplan.*

Lifshitz, Jaime. On the stability of principal periodic orbits in the theory of primary cosmic rays. J. Math. Phys. Mass. Inst. Tech. **21**, 284–292 (1942). [MF 8174]

This paper continues work on the orbits of cosmic ray particles in the earth's magnetic field, using the theory developed by Störmer, Lemaitre, Vallarta and others [see M. S. Vallarta, An Outline of the Theory of the Allowed Cone of Cosmic Radiation, Univ. of Toronto Studies, Appl. Math. Ser., no. 3, Toronto, 1938]. Störmer discovered two families of periodic orbits [Z. Astrophys. **1**, 237–274 (1930)], called "internal" and "external." Godart investigated the stability of the external orbits by characteristic exponents, using a new method which reduces the variational equations to one of Hill's type [Ann. Soc. Sci. Bruxelles (1) **58**, 27–41 (1938)]. In the present paper Godart's results are recalculated with more accuracy and the stability of 19 of the internal orbits is investigated by the method of the secular determinant, as Godart's method is inapplicable to this case. The following conclusions are obtained: the external orbits are unstable; the internal orbits are stable except for a certain intermediary range of Störmer's parameter. The fact, not usually emphasized, that the characteristic exponents are determined only up to a multiple of $i\alpha$, where $2\pi/\alpha$ is the period of the known solution, turns out to be of importance in the work. *W. Kaplan* (Ann Arbor, Mich.).

Chandrasekhar, S. and von Neumann, J. The statistics of the gravitational field arising from a random distribution of stars. II. The speed of fluctuations; dynamical friction; spatial correlations. Astrophys. J. **97**, 1–27 (1943). [MF 8176]

The authors assume, as in their first paper [Astrophys. J. **95**, 489–531 (1942); these Rev. **3**, 281; cf. also Chandrasekhar, Astrophys. J. **94**, 511–524 (1941); these Rev. **3**, 281], a stationary Poisson distribution of stars, with mutually independent Gaussian velocity distributions, independent of position. Let F be the gravitational force of the stars acting on a mass point P moving with velocity v. The conditional expectation and dispersion, given F and v, of dF/dt are calculated. The authors' previous paper had assumed $v=0$. The most interesting conclusion drawn from the results is the existence of "dynamical friction": there is a tendency for the moving point P to be decelerated in the direction of its motion, a tendency proportional to $|v|$.

J. L. Doob (Washington, D. C.).

Hydrodynamics, Aerodynamics

Lin, C. C. On the motion of a pendulum in a turbulent fluid. Quart. Appl. Math. **1**, 43–48 (1943). [MF 8186]

The author discusses the motion of a pendulum subjected to random forces, relating the correlation function of the pendulum displacements with that of the forces. He obtains Schumann's results [Philos. Mag. (7) **33**, 138–150 (1942); these Rev. **4**, 27] somewhat more rigorously. A still more direct and rigorous method would be to write the usual second order differential equation for the angular displacement, solve it explicitly and thus find at once the relation between the two correlation functions. This method has already been used frequently in Brownian movement problems, in which case the random forces at different times are uncorrelated. [Cf., for example, W. R. van Wijk, Physica **3**, 1111–1119 (1936).] *J. L. Doob.*

Schlichting, H. Die Grenzschicht an der ebenen Platte mit Absaugung und Ausblasen. Luftfahrtforschung **19**, 293–301 (1942). [MF 8272]

The paper contains a theoretical approach to the problems of boundary layer control by suction or ejection. The laminar and the turbulent flow in the boundary layer formed along a flat plate are studied under the assumption that the velocity component normal to the plate is a given function $v_0(x)$ of the distance x from the leading edge of the plate. For constant v_0 the thickness of the boundary layer is found to be asymptotically constant in the case of suction and proportional to x in the case of ejection. *W. Prager.*

Kochin, N. E. and Loytzansky, L. G. An approximate method of calculating the laminar boundary layer. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 262–266 (1942). [MF 8094]

This paper is concerned with the integration of the non-linear partial differential equations for two-dimensional

boundary-layer flow. The authors write the von Kármán integral relation in the form

$$\frac{df}{dx}=\frac{U''}{U'}f+\frac{U'}{U}F,$$

where $f(x)=U'\delta^{**2}/\nu$ and

$$F(x)=-4f-2U'\delta^{*}\delta^{**}/\nu+2\tau_0\delta^{**}/\mu U.$$

Here $U(x)$ is the velocity outside the boundary layer, δ^* and δ^{**} are the displacement and momentum thicknesses, τ_0 the shear stress at the wall and x, y, u, v, ν, ρ and μ have their usual meanings.

In the special case $U=cx^m$ (c and m being constants) an exact solution of the boundary layer is known; it appears in the form $u=U\Phi'(\xi)$, where $\xi=y(U'/\nu\beta)^{\frac{1}{2}}$, $\beta=2m/(m+1)$. In this case it is found that $\delta^*=(\nu\beta/U')^{\frac{1}{2}}A(\beta)$, $\delta^{**}=(\nu\beta/U')^{\frac{1}{2}}B(\beta)$ and $\tau_0=(\mu\rho U^2U'/\beta)^{\frac{1}{2}}\Phi''(0)$, and that f and F are independent of x, depending only upon β. From the results of Hartree [Proc. Cambridge Philos. Soc. **33**, 223–239 (1937)] the authors have calculated f, F, $A(\beta)$, $B(\beta)$ and $\Phi''(0)$ and tabulate them as functions of β.

Proceeding to the general case in which $U(x)$ is an arbitrarily given function, the authors write $u=U(x)\cdot\Phi'(\xi,\beta)$, where now $\beta=\beta(x)$. The functions of f and F then have the same forms as before in terms of β, $A(\beta)$, $B(\beta)$ and $\Phi''(0)$. The authors' approximation consists of assuming that F, $A(\beta)$, $B(\beta)$ and $\Phi''(0)$ bear the same relationships to f as in the special case above. This device permits integration of the von Kármán equation to find $f(x)$, and thereby provides a complete determination of δ^*, δ^{**}, τ_0, etc. as functions of x. As an example they carry out the calculations for the case $U=U_0\cdot(1-x)$, which has been treated by Howarth [Proc. Roy. Soc. London. Ser. A. **164**, 547–579 (1938)]. The integration of the von Kármán relation is carried out by a simple approximate procedure. The result for the value of x at separation ($x=0.106$) is in satisfactory numerical agreement with Howarth's accurate value ($x=0.12$). *W. R. Sears* (Inglewood, Calif.).

Kochin, N. E. On the steady oscillations of an aerofoil of round shape in plane. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 287–316 (1942). (Russian. English summary) [MF 7583]

[The actual English translation of the title is "Steady oscillations of an airfoil with a projection of circular form."] The author considers a slightly cambered airfoil whose projection on the x, y-plane is a circle S of radius a. The airfoil moves uniformly in the direction of the positive x-axis. In addition, small vibrations of the airfoil of frequency ω are assumed. The equation of the airfoil can be written

$$z=\zeta_0(x,y)+\zeta_1(x,y)\cos\omega t+\zeta_2(x,y)\sin\omega t,$$

where ζ_k/a, $\partial\zeta_k/\partial x$, $\partial\zeta_k/\partial y$ are small. Behind the airfoil a discontinuity strip Ξ of width a extending to infinity is assumed. The potential ϕ can be written in the form

$$\phi=\phi_0(x,y,z)+\Re[\Phi(x,y,z)\exp(-i\omega t)],$$

where ϕ_0 and $\Phi=\phi_1+i\phi_2$ are harmonic functions which are regular in the whole space except on $S+\Xi$. Thus the problem is reduced to that of determining harmonic functions ϕ_0, Φ which are regular outside of $S+\Xi$ and which satisfy the boundary conditions: $\partial\phi_0/\partial z=Z_0(x,y)$, $\partial\Phi/\partial z=Z(x,y)$ on S, $\partial\phi_0/\partial x=0$, $\partial\Phi/\partial x+ik\Phi=0$ on Ξ. Here Z_0 and Z are given functions. Furthermore, we have $\lim_{x\to\infty}\partial\Phi/\partial x=\lim_{x\to\infty}\partial\Phi/\partial y=\cdots=0$. In his previous paper [J. Appl. Math. Mech. **4**, 3–32 (1940)] the author gave a representation for harmonic functions which are regular in the exterior of $S+\Xi$, and found an expression for the function ϕ_0. Now

$$\Phi(x,y,z)=\iint_S f(\xi,\eta)L(x,y,z;\xi,\eta)d\xi d\eta,$$

where L is a complicated (but known) function and $f(x,y)$ is determined by this integral equation. Finally the author shows that the solution has the following property: the speed of the fluid at the trailing edge is finite; the speed at the leading edge can become infinite as $\delta^{-\frac{1}{2}}$, where $\delta=\delta(x,y,z)$ denotes the distance of the particle from the leading edge. Using this formula the author obtains, by elaborate computation, comparatively simple expressions for the pressure distribution, for lift and drag and for the moment. A numerical example (an airfoil of the above form which periodically changes its angle of attack) is given. *S. Bergman* (Providence, R. I.).

Theilheimer, F. The influence of sweep on the spanwise lift distribution of wings. J. Aeronaut. Sci. **10**, 101–104 (1943). [MF 8126]

The author considers a lifting line with sweepback and proceeds to calculate, for a given distribution of circulation over the span, the effect of the sweepback on the induced velocity at the line. The formulae are similar to those of Crean [Aircraft Engrg. **10**, 245–247 (1938)], but an obvious error in the earlier paper has been corrected. The problem of the infinite induced velocities that result at the center of the wing, in violation of the assumption of the method, is not discussed. As a numerical example the induced-velocity increment is calculated for a certain lift distribution for sweepback of 15° and 30°, respectively. In this calculation the trailing-vortex system is approximated by eight concentrated vortices on each semispan. *W. R. Sears.*

Randels, W. C. A new derivation of Munk's formulae. Quart. Appl. Math. **1**, 88–92 (1943). [MF 8192]

In this note the author derives the Munk integral formulae for the lift and moment of a thin cambered airfoil in two-dimensional steady flow by an application of the method of the acceleration potential. The problem is that of calculating certain integrals involving the tangential velocity component when the normal component is given at the airfoil; the Poisson integral is employed after a simple conformal transformation. [See also H. J. Stewart, J. Aeronaut. Sci. **9**, 452–456 (1942); these Rev. **4**, 121.] *W. R. Sears* (Inglewood, Calif.).

Simonov, L. A. Construction of shapes by means of a hodograph of velocities. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 193–222 (1941). (Russian. English summary) [MF 7707]

This is a continuation of a previous investigation [same J. **4**, 97–116 (1940); these Rev. **3**, 92]. The author studies hodographs of incompressible fluid flows around an airfoil. Let the first axis of the profile coincide with the positive real axis, and let v_1, v_2 and v_α be the speeds at some point on the profile arising from flows with the angles of attack $\pi/2$, 0 and α, respectively. The speed at infinity is taken to be the same in each case. Then $(v_\alpha/v_1)=\sin\alpha-\sin(\alpha-\lambda)$, $(v_2/v_1)=-\sin\lambda$, where λ depends on the point chosen. This result enables the author to give a procedure for determining the speed distribution on the boundary of the profile from a given speed distribution v_1, that is, of a purely circulatory flow. He gives an analytic expression for the complex poten-

tial considered as a function of the velocity which leads to purely circulatory flows around obstacles having (in the physical plane) shapes of airfoils. The above-mentioned expression depends upon several parameters. The author analyzes the influence of the variation of the parameters on the shape of the airfoil. The investigation is illustrated by examples in which the velocity distribution is determined for different values of the angle of attack. *S. Bergman.*

Mokrzycki, G. A. The best turning of an aircraft. Bull. Polish Inst. Arts Sci. Amer. **1**, 646–650 (1943). [MF 8304]

Frössel, W. Berechnung der Reibung und Tragkraft eines endlich breiten Gleitschuhes auf ebener Gleitbahn. Z. Angew. Math. Mech. **21**, 321–340 (1941). [MF 7659]

The problem is to determine the velocity distribution in a viscous lubricant between a rectangular slider and a flat plate when the slider is moving with constant velocity, relative to the plate, in a direction parallel to the plate. Two cases are considered: (a) the sliding face of the slider is flat but slightly inclined to the plate and (b) the sliding face is a portion of a cylindrical surface of large radius. The Navier-Stokes equations are linearized and made two dimensional by neglecting the product terms and the terms containing derivatives in a direction parallel to the plate and perpendicular to the motion. The resulting equation for the pressure has in the case (a) the form $\Delta p+ax^{-1}p_x+bx^{-2}=0$, where a and b are constants; for case (b) the coefficients are slightly more complicated. These equations are solved by a separation of variables. The resulting ordinary differential equations in x are solved in terms of Bessel functions in (a) and integrated numerically in (b). Tables of the first four to six eigenvalues and corresponding eigenfunctions and their derivatives are given for several particular shapes of slider, in all of which the width is greater than the length (dimension in the direction of motion). Drawings of the surfaces showing pressure distribution and frictional force distribution are given in typical cases. The dependence of total load and total resistance on the shape of slider is also studied.

P. W. Ketchum (Urbana, Ill.).

Jeffreys, Harold. A derivation of the tidal equations. Proc. Roy. Soc. London. Ser. A. **181**, 20–22 (1942). [MF 7256]

In the derivation of the tidal equations a difficulty arises from the fact that when the ellipticity is taken into account the three directions of co-latitude, longitude and height are not strictly orthogonal. The author simplifies the discussion by using a set of exactly orthogonal coordinates.

A. Weinstein (Toronto, Ont.).

Haurwitz, B. The effect of a gradual wind change on the stability of waves. Ann. New York Acad. Sci. **44**, 69–80 (1943). [MF 8434]

Kalinske, A. A. Turbulence and the transport of sand and silt by wind. Ann. New York Acad. Sci. **44**, 41–54 (1943). [MF 8433]

Riesenkampf, B. K. and Kalinin, N. K. Dreidimensionale Grundwasserbewegung mit einer freien Oberfläche von der Form eines Ellipsoides. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 283–286 (1941). (Russian. German summary) [MF 7713]

As it is known, the potential function ϕ of irrotational, steady motion of an incompressible fluid around an ellipsoid is given by $\phi=v_\infty\sum_{k=1}^{3}(1+A_k)\lambda_k x_k$, where the A_k are constants and the λ_k the direction cosines of the velocity vector at infinity. The authors indicate that, by a suitable choice of the constants A_k, a surface $(\partial\varphi/\partial n)=0$ becomes a free surface (that is, a surface on which the pressure is constant). Therefore, the flow can be considered as a motion of ground waters with ellipsoidal "free surface." *S. Bergman.*

Gantmacher, F. R. and Segal, B. I. A method of hydromechanical design of a system of dams. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 94–99 (1942). [MF 7613]

The new system of reinforced concrete dams designed by the authors is based upon the insertion of intermediate waterlevels between the headwater and the tailwater level, calculated to decrease the forces acting upon the structure. The boundary conditions for seepage under such a dam are obviously very complicated and none of the usual approximative methods of seepage determination is applicable. A method based upon conformal mapping is worked out; for a special case of the new structure the results are evaluated numerically. *P. Neményi* (Fort Collins, Colo.).

Rossbach, H. Über die unter einem Damm durch eine horizontale Parallelschicht sickernde Wassermenge und die Auftriebsdruckverteilung an der Dammbasis. Z. Angew. Math. Mech. **22**, 65–71 (1942). [MF 7665]

The plane steady flow of ground water through a homogeneous aquifer underlying a dam is investigated by aid of the function of a complex variable and of conformal mapping. The quantity of seepage water as a function of the thickness of the aquifer is evaluated. Also the piezometric pressure distribution on the bottom of the dam is determined. *P. Neményi* (Fort Collins, Colo.).

Fortier, André. Sur le calcul graphique des intumescences de hauteur finie. C. R. Acad. Sci. Paris **214**, 710–712 (1942). [MF 7887]

The author's analytic-graphic methods for the study of waves traveling along canals or rivers is extended to such cases in which the wave height is large compared to the undisturbed water depth in the canal, but the travel-speed of the surface shape is small. *P. Neményi.*

Eksergian, Rupen. The fluid torque converter and coupling. J. Franklin Inst. **235**, 441–478 (1943). [MF 8345]

The author gives a theory of a machine, which through the medium of a fluid transforms a power given as a certain torque times a certain angular speed into another torque and speed. An elementary hydraulic theory is developed in great detail. In an appendix some fundamental relations are derived with the more advanced methods of fluid mechanics. The damping effect of a fluid coupling (or hydraulic clutch) in a torsional vibration system is also analyzed. Two examples of vibrational problems arising in applying the clutch into systems with transmission gears and rotational inertias are worked out. The June issue of the Journal of the Franklin Institute published a slip with errata of the paper. *I. Opatowski* (Chicago, Ill.).

Theory of Elasticity

Ratzerdorfer, J. Determination of the buckling load of struts with successive approximations. J. Royal Aeronaut. Soc. **47**, 103–105 (1943). [MF 8237]

The solution of $EI(u)d^2v/du^2+Pv+M=0$ with $v(0)=v(1)=0$ is approximated by the solution of the correspond-

ing difference equation. The latter is solved in the usual manner. *W. Feller* (Providence, R. I.).

Weber, Constantin. Halbebene mit periodisch gewelltem Rand. Z. Angew. Math. Mech. **22**, 29–33 (1942). [MF 7673]

Airy's stress function is determined for a half plane with periodic wave boundary. The method depends on the function $z=\zeta+i\sum a_n e^{in\zeta}$, a_n real, which maps such a half plane onto a half plane with straight boundary. As an example the stress function is determined explicitly for a half plane with boundary which is straight except for a succession of semicircular notches of uniform size and spacing, the spaces between the notches being equal to the diameter of the notches. It is found that the maximum stress concentration in an infinite plate, under uniform tension parallel to the edge of the plate, due to such a row of notches along the edge of the plate is 2.13 fold, compared with 3.07 fold if there is only a single notch. *P. W. Ketchum.*

Dolberg, M. The deflection of bars under compression. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 369–374 (1942). (Russian. English summary) [MF 7587]

The author studies the deflection of a bar under an axial load P. Under usual assumptions, the problem of determining the deflection $\delta y(x)$ leads to the determination of eigenfunctions $y_n(x)$ of an ordinary differential equation of the fourth order. Using the principle of the minimum of potential energy, the above-mentioned equation can be replaced by an integral equation. Using certain relations involving Green's functions of ordinary differential equations of the fourth order and the theory of integral equations, the author gives an approximate formula for determining δ_n as a function of P. *S. Bergman.*

Reutter, F. Eine Anwendung des absoluten Parallelismus auf die Schalentheorie. Z. Angew. Math. Mech. **22**, 87–98 (1942). [MF 7663]

A brief discussion is given of the parallel displacement of a vector as given originally by Levi-Civita. This is applied to the deformations of a thin shell under the action of forces, special attention being paid to the Mittelfläche of the shell and to the equilibrium conditions. The simplified methods of tensor analysis are neglected and all equations are written out at length. *T. Y. Thomas.*

Reissner, Eric. Note on the expressions for the strains in a bent, thin shell. Amer. J. Math. **64**, 768–772 (1942). [MF 7178]

In a previous paper [Amer. J. Math. 63, 177–184 (1941); these Rev. 2, 272] the author discussed quite generally the linear theory of bending of thin shells. In the present note he points out that earlier writers on the subject have usually been inconsistent in neglecting certain terms which are of the same order as those retained. The complete expressions for the relevant quantities are given and compared with those of a number of other authors for the case of the cylindrical shell. The author notes that the terms commonly neglected are not obviously of negligible influence on the results of calculations. *J. J. Stoker.*

Reissner, Eric. Note on some secondary stresses in thin-walled box beams. J. Aeronaut. Sci. **9**, 538–542 (1942). [MF 7558]

The author is interested in discussing certain stresses developed in the thin sheets (that is, thin elastic plates) which cover the top and bottom of box beams. In the simple beam theory it is assumed, roughly speaking, that all longitudinal fibers of the beam are bent into curves parallel to that of the centroidal axis. It is readily seen that this assumption does not hold for the kind of beams considered by the author. A rational method of making a correction for this inaccuracy is developed by the author. The results of a complete numerical solution in a special case are given; in particular, certain of the "secondary" stresses are found to have magnitudes which are an appreciable fraction of the primary stresses. *J. J. Stoker* (New York, N. Y.).

Fortin, René. Étude analytique et graphique de la poutre continue. Rev. Trimest. Canad. **29**, 57–87 (1943). [MF 8153]

Review of some analytical and graphical methods of analysis of the stresses in continuous beams (theorem of three moments, method of fixed points). *W. Prager.*

Papkowitsch, P. F. Zwei Fragen zur Theorie der dünnen elastischen Platten. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 359–374 (1941). (Russian. German summary) [MF 7720]

Applying the method of Galerkin, the author studies the buckling of plates of two forms with various boundary conditions. (I) The plate is of the form $0<\theta<\alpha$, $a<r<b$, where r, θ are polar coordinates. The author writes the solution in the form $w=f(r,\theta)+\sum\phi_k(r)\sin k\pi\theta/\alpha$, where f is an arbitrary solution of the nonhomogeneous equation $\Delta\Delta w=q(x,y)$, q representing the exterior forces

$$\phi_k(r)=a_kr^{\alpha_k}+b_kr^{-\alpha_k}+c_kr^{\alpha_k+2}+d_kr^{-\alpha_k+2},\quad \alpha_k=k\pi\alpha^{-1}.$$

The above expression is the general solution if $w=M_\theta=0$ on $\theta=0$ and on $\theta=\alpha$. The constants $a_k, b_k, \cdots$ are determined from the boundary conditions. The author also indicates the general solution for arbitrary boundary conditions. (II) The plate has the form of a rectangle. Now the author assumes that the solution for different boundary conditions is given in one of the forms

$$w=\sum f_k(y)\sin k\pi x/a,$$
$$w=\sum f_k(y)\cos k\pi x/a,$$
$$w=\Re\{Y_k(y)[a_ke^{-\lambda_kx}+b_ke^{-\lambda_k(a-x)}]\};$$

here the Y_k satisfy an ordinary differential equation of the fourth order. The author discusses how to determine f_k, Y_k and the constants a_k, b_k from the boundary conditions. In particular, in the case of the functions Y_k, he shows that the latter functions satisfy certain generalized orthogonal relations, which enables him to determine the constants a_k and b_k in a simple way. *S. Bergman* (Providence, R. I.).

Levy, Samuel. Buckling of rectangular plates with built-in edges. J. Appl. Mech. **9**, A-171–A-174 (1942). [MF 7474]

The problem of finding the buckling load of a thin homogeneous rectangular plate with clamped edges is treated for a compressive force acting normal to one pair of opposite edges. Employing a double Fourier development for the deflection function, the author shows how to obtain the critical buckling loads by evaluating the necessary determinants which insure the solutions of a doubly infinite set of homogeneous linear equations. Summation formulas ascribed to A. Sommerfeld [probably due to integral equation theory of Kneser] yield closed expressions which expedite the calculations. Estimated values of the infinite

determinants are obtained by extrapolating from the successive approximations calculated from finite forms of the determinant. Results are listed for plate ratios a/b from 0.75 to 4.00 and comparisons made with earlier writers.
D. L. Holl (Ames, Iowa).

Stevenson, A. C. The boundary couples in thin plates. Philos. Mag. (7) **34**, 105–114 (1943). [MF 8132]

Following the pattern of a previous paper [Philos. Mag. (7) **33**, 639–661 (1942); these Rev. **4**, 63], the author makes use of a complex variable to transform Cartesian components of stress resultants and stress couples in a thin plate to curvilinear coordinates. A discussion of a complex potential function for a plate, whose boundary is determined by the mapping of the unit circle $|\sigma|=1$ by means of $z=c\sigma(1+\lambda\sigma^n)$, $c>0$ and $\lambda>0$, is given.
I. S. Sokolnikoff (Madison, Wis.).

Panov, D. J. On large deflections of slightly corrugated circular membranes. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 303–318 (1941). (Russian. English summary) [MF 7715]

This investigation of the deflection of corrugated circular membranes subjected to a hydrostatic load is based on the thin shell formulas given in Love's treatise on elasticity. An approximate formulation of the problem provides an explanation of an experimental observation that the deflection of the membrane depends on the side to which the load is applied. An explanation of other known experimental facts does not appear in the results of this paper, but they presumably can be deduced by improving the degree of approximation in the differential equations. This will require very heavy calculations. The paper should prove of interest to a designer of aneroidal instruments.
I. S. Sokolnikoff (Madison, Wis.).

Parkhomovsky, J. On a method of approximate solution of the problem of torsion. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 89–91 (1942). [MF 8010]

The approximation method used is the following. The torsion problem is solved exactly for a certain contour, for example, that given by $r=f(\vartheta)$ in polar coordinates. One considers then the modified contour $r=f(\vartheta)+\epsilon g(\vartheta)$, ϵ being a certain constant. The solution is then developed with respect to ϵ and terms of order ϵ^2 and higher are neglected. The author gives the results for (1) a contour which differs little from the circle, (2) a contour resembling the cross section of a propeller and which results from taking an ellipse as original contour and (3) a region differing slightly from the circular ring. *J. J. Stoker* (New York, N. Y.).

Kilchevsky, N. A. On axial-symmetric deformations and elastic stability of circular tubes under the action of longitudinal compressing forces. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 497–508 (1942). (Russian. English summary) [MF 8364]

The present paper examines the causes of divergence in the results obtained through experiment and by theoretical investigations of the stability of cylindrical shells subjected to compressing forces directed along the generatrix, in cases of axial-symmetric deformation. Critical stresses in freely-supported tubes are considered. New boundary problems are dealt with in the stability of a tube with freely sliding ends and the stability of a tube with one clamped and the other a freely sliding end. As a result of this investigation, it is shown that critical stresses in a tube with freely-supported ends generally speaking always exceed the limits of stability, and that the collapsing of the tube may take place long before the load attains its critical value.
Author's summary.

Stevenson, A. C. The torsion of a fluted column. Philos. Mag. (7) **34**, 115–120 (1943). [MF 8133]

The author considers the boundary given by $u=0$ in the w-plane, where $w=u+iv$, $z=ce^w(1+ke^{nw})$, c and k constants. The coefficients are calculated in a general type of series solution obtained by Rosa Morris and proved here by a method of residues. Diagrams are given showing the forms of boundary for the cases $n=1$, $k=\frac{1}{8}$; $n=12$, $k=\frac{1}{16}$. Special attention is given to the stress at the bottom of a groove.
H. Bateman (Pasadena, Calif.).

Sokolovsky, W. On the allowance for strain hardening of material in the problem of elastic-plastic torsion. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 46–51 (1942). [MF 7628]

The author investigates the manner in which a small amount of strain hardening influences the solution of the problem of the elastic-plastic torsion of cylindrical bars. In the plastic region the stress-strain relations are adopted in the form proposed by H. Hencky: $\gamma_x=(1+\varphi)\tau_x/G$, $\gamma_y=(1+\varphi)\tau_y/G$, where τ_x, τ_y are the components of the shearing stress in the cross-sectional planes of the bar, γ_x, γ_y the corresponding components of the deformation and G is the modulus of rigidity. The function $\varphi(x, y)$ may be eliminated by means of the yield condition $\tau=k+\lambda G_\gamma$, where $\tau=(\tau_x^2+\tau_y^2)^{\frac{1}{2}}$, $\gamma=(\gamma_x^2+\gamma_y^2)^{\frac{1}{2}}$ and k, λ are given constants. The function $\varphi(x, y)$ and the angle ψ between the stress vector and the x-axis are expressed as power series in the parameter λ and the differential equations for the first two terms of these series are established. For moderate strain hardening (small λ) these terms may be taken as indications for the manner in which the strain hardening tends to influence the stress distribution. Three examples are treated, two of which are trivial (circle and infinite strip as cross sections). Though not presented in this manner, the very interesting third example is obviously constructed by starting from a simple solution of the elastic torsion problem and determining the (closed) curve C along which the yield condition is fulfilled. This curve is taken as the boundary between the elastic and the plastic region; the stresses and deformations on the plastic side of C are known by continuity and the plastic solution in the exterior of C corresponding to these boundary conditions is constructed. Any closed curve whose tangent has everywhere the direction of the stress vector can be taken as the contour of the cross section. In the case considered in the paper a contour of an elliptic type is obtained.
W. Prager (Providence, R. I.).

Sokolovsky, W. Fundamental equations of the limit equilibrium of earthy medium. C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 47–51 (1942). [MF 7597]

Sokolovsky, W. On the stability of slopes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 71–77 (1942). [MF 7599]

Continuing the work outlined in two earlier notes [C. R. (Doklady) Acad. Sci. URSS (N.S.) **22**, 153–157 (1939); **23**, 14–16 (1939)], the author treats the plane problem for an earthy medium, the limit equilibrium of which is characterized by Coulomb's condition. The coefficient of cohesion and the angle of friction are assumed to be given functions

of the coordinates. A line on which the derivatives of any stress component with respect to the coordinates become infinite is called a line of rupture. In the first paper the author establishes the condition for the presence of a line of rupture and states that the envelope of the slip lines is the line of rupture. The mathematical analysis is modeled on the pattern of S. Christianovitch's treatment of the plane problem of plasticity [Rec. Math. [Mat. Sbornik] **1** (**43**), 511–534 (1936)]. The second paper deals with the following plane problem: The earthy medium under consideration is supposed to be bounded by a cylindrical surface with horizontal generators (the slope) and a horizontal plane. The pressure acting on the horizontal plane is a given function of the distance from the line of intersection of this plane and the slope. The slope is supposed to be free from stresses and its form in the state of limit equilibrium is to be determined.

W. Prager (Providence, R. I.).

Kachanov, L. M. Plastic-elastic state of solids. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 431–438 (1941). (Russian. English summary) [MF 7725]

The paper is concerned with the thermodynamics of solids characterized by the following conditions: (1) the state of the solid is completely defined by a temperature t and the strain tensor ϵ_{ik}; (2) the mean normal strain $\epsilon=\epsilon_{ii}/3$ depends on the mean normal stress $\sigma=\sigma_{ii}/3$ (σ_{ik}=stress tensor) and on the temperature t according to $\epsilon=k\sigma+\alpha t$, where k and α are constants; (3) the strain deviation $e_{ik}=\epsilon_{ik}-\epsilon\delta_{ik}$ and the stress deviation $s_{ik}=\sigma_{ik}-\sigma\delta_{ik}$ are related by $e_{ik}=\psi s_{ik}$, where the factor ψ may depend on the scalar invariants of stress and strain. From (3) it follows that $E=(e_{ik}e_{ik})^{\frac{1}{2}}$ and $S=(s_{ik}s_{ik})^{\frac{1}{2}}$ are related by $E=\psi S$. Applying the principles of thermodynamics, the author proves that one of the following three conditions must be fulfilled: (a) ψ=const., (b) S=const., (c) $\psi=\psi(S)$. The condition (a) characterizes the elastic body, (b) leads to Hencky's theory of plasticity and (c) is said to represent strain hardening. [As a theory of plasticity this theory is open to the same criticism as that of Hencky. The state of a plastic solid is not defined by t and ϵ_{ik}; the stresses σ_{ik} depend not on the final deformation ϵ_{ik} alone, but on the complete history of deformation. The solids studied in the present paper would more correctly be described as solids of nonlinear elasticity.] Considering, in particular, isothermal deformations of the solid (b), the author eliminates the stress components between the equilibrium conditions, the "yield condition" S=const. and the stress strain relations, thus obtaining three nonlinear differential equations for the components of the displacement vector. The deformation with spherical symmetry (for example, spherical shell under interior or exterior pressure) is treated as an example. *W. Prager* (Providence, R. I.).

Arnold, Lee. Vector solution of the three-degree case of wing bending, wing torsion, aileron flutter. J. Aeronaut. Sci. **9**, 497–500 (1942). [MF 7419]

The flutter speed V and the flutter frequency ω for three-degrees of freedom cases are determined by the condition that a certain third order determinant with complex elements involving V and ω vanishes [Th. Theodorsen, Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 496, 1934]. To facilitate the laborious evaluation of this condition and thereby to effect considerable time saving the author has developed a graphical procedure making use of the property of complex numbers to permit geometrical addition.

E. Reissner (Cambridge, Mass.).

Flax, Alexander H. Three-dimensional wing flutter analysis. J. Aeronaut. Sci. **10**, 41–47 (1943). [MF 7976]

The starting point of the developments of this paper is the assumption that a wing may be treated as a cantilever beam possessing given torsional and bending characteristics plus an aileron the elasticity of which is concentrated along its hinge line. Every spanwise element of the wing is acted upon by air forces which on the basis of a neglection of the aerodynamic span effect are known linear homogeneous functions of the local deflections. Assuming the shape of the deflection curves in the three modes, the principle of virtual work leads to three linear homogeneous equations for the three amplitude factors. Vanishing of the (complex) determinant of this system determines flutter speed and flutter frequency. The author points out that thereby is obtained a rational procedure for the determination of the structural parameters of a substitute two-dimensional system. More general and thereby less simple procedures of a similar nature have previously been given by S. Loring [S.A.E. J. **49**, 345–356 (1941)] and H. M. Bleakney [J. Aeronaut. Sci. **9**, 56–63 (1941)]. *E. Reissner* (Cambridge, Mass.).

Hoff, N. J. Stresses in space-curved rings reinforcing the edges of cut-outs in monocoque fuselages. J. Royal Aeronaut. Soc. **47**, 35–83 (1943). [MF 8069]

As a first step toward an analysis of the stresses in the neighborhood of cut-outs in monocoque fuselages, the author studies the stresses in the skew rings used to reinforce the edges of such cut-outs. The general procedure (to be adopted in the case of a ring of arbitrary shape) is indicated, but only the ring with two orthogonal planes of symmetry is treated in detail. With a view to investigating the influence of the orientation of the principal axes of inertia of the cross sections, the following cases are studied: (a) the principal axes of inertia coincide with the normal and binormal of the center line of the ring; (b) one principal axis of inertia coincides with the normal of the surface of the fuselage; (c) one principal axis of inertia possesses a fixed direction (for example, that of the axis of the fuselage). Special attention is given to the effects of nonuniform warping of thin-walled sections. An appendix contains a detailed account of the theory of torsion of thin-walled open sections. *W. Prager* (Providence, R. I.).

Bronsky, A. P. Residual effect in rigid bodies. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 31–56 (1941). (Russian. English summary) [MF 7700]

This paper, which is partly expository in character, is devoted to the study of elastic residual effects and relaxation. Using the theory of Boltzmann and the methods of Volterra for integro-differential equations the author investigates two problems: the case when the stresses within the body are constant and the case when the deformations are constant. On the basis of experiment new expressions for the coefficients of residual effect and relaxation are obtained from which values of well-known coefficients given by Maxwell, Boltzmann, and others follow. In the conclusion, problems are solved in which the phenomenon of elastic residual effect is taken into consideration. In particular, the problem of the distribution of stresses over the contact area is carried to completion. *S. Bergman* (Providence, R. I.).

Prager, W. and Hay, G. E. On plane rigid frames loaded perpendicularly to their plane. Quart. Appl. Math. **1**, 49–60 (1943). [MF 8187]

This paper points out an analytical analogy between certain plane structures loaded in their plane and certain dual plane structures loaded perpendicularly to their plane. It is shown that this analogy permits the immediate determination of the stress resultants in a given statically determinate frame if the displacemenets of the dual frame are known. A procedure dual to the column analogy of H. Cross [Univ. Illinois Eng. Exp. Station, Bull. no. 215, Urbana, Ill., 1930] is also outlined for the determination of the stress resultants in an indeterminate frame loaded perpendicularly to its plane. The relationships explicitly presented apply to polygonal frames having not more than two members intersecting at a point, and include as a special case the method of conjugate beams proposed by H. M. Westergaard [J. Western Soc. Engr. **26**, 369–396 (1921)].

F. B. Hildebrand (Cambridge, Mass.).

Timoshenko, Stephen P. Theory of suspension bridges. J. Franklin Inst. **235**, 213–238 (1943). [MF 8139]

This paper discusses various methods of analysis of suspension bridges. The first two sections treat the cases of a perfectly flexible cable and of an unstiffened suspension bridge. For these cases equations for the calculation of deflections and changes in cable tension produced by a live load are developed. In the third section the fundamental equations for stiffened suspension bridges are derived. Finally an analysis of a single span stiffened suspension bridge is given. *A. E. Heins* (Cambridge, Mass.).

Timoshenko, Stephen P. Theory of suspension bridges. II. J. Franklin Inst. **235**, 327–349 (1943). [MF 8216]

This article is a continuation of the paper reviewed above. There is a discussion of the use of trigonometric series in the calculation of deflections. Such series simplify the calculations and improve the accuracy. The author treats as one example the case of one concentrated force acting on a stiffening truss. This is followed by a treatment of the three span suspension bridges with simply supported and continuous stiffening trusses. The paper concludes with a discussion of stiffening trusses with variable cross section. Numerical examples are given.

A. E. Heins (Cambridge, Mass.).

Reissner, Hans. Oscillations of suspension bridges. J. Appl. Mech. **10**, A-23–A-32 (1943). [MF 8045]

This paper discusses the free and self-oscillations of stiffened cable systems. For the case of free oscillations, the author is concerned with a system of linear differential equations with variable coefficients. Such quantities as horizontal displacements, inertia forces, change of thrust along the cable, the slope of the cable and the torsional stiffness of the girder are brought into the calculations. These differential equations are combined into a form which will permit a direct application of the Rayleigh-Ritz method. The second part of the paper deals with self excited oscillations and has been prepared under the influence of observations, experiments and conclusions reached in connection with an official report on the Tacoma bridge failure.

A. E. Heins (Cambridge, Mass.).

Jeffreys, Harold. On pulses whose travel times are not true minima. Proc. Cambridge Philos. Soc. **39**, 48–51 (1943). [MF 7968]

The author states that his experience in the construction of seismological tables would seem to indicate that there are two distinct types of paths followed by earthquake waves, brachistochronic paths and nonbrachistochronic paths. In the case of the former type of path, for which a true minimum travel time exists, the statistical residuals show a pattern characterized by a frequency curve with a sharp peak, whereas, in the case of the second type of path, for which the travel times are stationary but have no true minimum, there is only a vague concentration of residuals spread over about half a minute. The present article is an account of the author's attempt to represent these conditions for the transmission of seismic pulses mathematically. The discussion is mathematical and cannot be adequately abstracted, but must be read in the original.

J. B. Macelwane, S. J. (St. Louis, Mo.).

MATHEMATICAL PHYSICS

Bhatnagar, P. L. and Kothari, D. S. A note on the principle of adiabatic invariance. Indian J. Phys. **16**, 271–275 (1942). [MF 8233]

In order to have more examples of simple classical systems having adiabatic invariants, the authors undertake the detailed study of the compound pendulum, the oscillating magnet and an electric oscillating circuit. The methods are the familiar ones in this subject, reposing on averages and approximations. *B. O. Koopman* (New York, N. Y.).

Rocard, Yves. Dualité des mécanismes d'autooscillation. C. R. Acad. Sci. Paris **214**, 601–603 (1942). [MF 7893]

From the mathematical result of this note incorrect physical conclusions are drawn. Forgetting that passive elements lead to positive coefficients in the integrodifferential equations, the author concludes from Routh's criteria for stability that in addition to the known way in which a system in two dependent variables can oscillate freely there exists another way. Consideration of the details omitted from the note shows that the new way requires negative energy-storing elements. In spite of this, the author appears to give two simple physical examples of this new way for positive-element systems to self-oscillate. He draws the conclusion, which he considers surprising, that with the new way the presence of positive resistance augments the instability giving rise to the oscillation. For positive-element systems this would indeed be surprising from an energy standpoint. *J. L. Barnes* (Princeton, N. J.).

Gutton, Henri et Ortusi, Antoine. Sur le rendement maximum d'un projecteur d'ondes. C. R. Acad. Sci. Paris **214**, 736–738 (1942). [MF 7885]

Consider an aperture A in the plane screen $z=0$. Assume (1) $\vec{e}_s=\vec{h}\phi(x, y)e^{i\omega t}dS$ is the field due to the Huygens secondary sources with the polarization direction $\vec{h}$ constant and parallel to the screen. Write (2) $\vec{e}_M=\vec{e}_s f(x, y; M)$, where $M=(x_0, y_0, z_0)$ is some fixed point and $f(x, y; M)$ is, of course, known. The writer indicates the physical significance of the simple problem of maximizing $\iint_A \phi f dS$ subject to $\iint_A \phi^2 dS=W_1$ a constant. Obviously the formal result is

$$\phi=f\left[W_1\Big/\iint_A f^2dS\right]^{\frac{1}{2}}.$$

D. G. Bourgin (Urbana, Ill.).

Tichonov, A. N. and Turkisher, R. J. The influence of an intermediate layer in vertical electric soundation. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1942**, 219–227 (1942). (Russian. English summary) [MF 8087]

The method of electric sounding is based on the exploration of the electric field due to a point source placed above the medium to be examined. For the interpretation of the data obtained the medium in question is usually assumed to consist of homogeneous horizontal layers of constant (but different) conductivity. The authors consider the following refined problem. A stratified medium is assumed to consist of an upper layer of thickness h_1 and conductivity σ_0, a lower layer with conductivity $\sigma=0$ and an intermediate layer of thickness h_2 whose conductivity varies linearily from σ_0 to 0. The problem of finding the stationary current distribution of a point source leads to a differential equation which can be separated in cylindrical coordinates. The boundary conditions uniquely determine a solution, which is expressed as a definite integral involving Bessel functions of orders 0 and 1. *V. Bargmann.*

Tichonov, A. N. The effect of inhomogeneity of earth crust on a field of telluric currents. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1942**, 207–218 (1942). (Russian. English summary) [MF 8086]

For geological and geophysical applications it is important to know how the distribution of electric currents is influenced by the inhomogeneity of earth's crust. In the present paper the author deals with a related, though simplified, problem. An infinite homogeneous cylinder of radius a and resistivity ρ_1, whose axis is parallel to the z-axis of a rectangular coordinate system, is imbedded in a homogeneous half space of resistivity ρ_2 bounded by the plane $y=0$. The author first considers a stationary current distribution which, for large values of y, asymptotically approaches constant values, the x-component being equal to I_0, while the y- and z-components vanish. Thus the problem is reduced to a two dimensional one which is treated as follows. The half plane $y\geqq 0$ is conformally mapped onto a circle with radius R_1 such that the cross section of the cylinder is transformed into a concentric circle of radius R. If the axis of the cylinder has the distance H from the bounding plane $y=0$, the conformal mapping is given by

$$\zeta=iR_1\{ih-(x+iy)\}\{ih+(x+iy)\}^{-1}$$

with

$$h^2=H^2-a^2,\quad aR=R_1(H-h).$$

Then the electric potential u satisfying the necessary boundary conditions can be expressed as the real part of a function u^*, where

$$u^*(\zeta)=\alpha\sum_{n=0}^{\infty}\beta^n U_0^*\left(\left(\frac{R}{R_1}\right)^{2n}\zeta\right) \qquad \text{for } |\zeta|\leqq R,$$

$$=\sum_{n=0}^{\infty}\beta^n\left\{U_0^*\left(\left(\frac{R}{R_1}\right)^{2n}\zeta\right)+\beta\bar{U}_0^*\left(\left(\frac{R}{R_1}\right)^{2n}\frac{R^2}{\bar{\zeta}}\right)\right\}$$

$$\text{for } R\leqq|\zeta|\leqq R_1,$$

with

$$U_0^*=\rho_2I_0iH\{1+i\zeta/R_1\}\{1-i\zeta/R_1\}^{-1},$$

$$\alpha=2\rho_1/(\rho_1+\rho_2),\quad \beta=(\rho_1-\rho_2)/(\rho_1+\rho_2).$$

Diagrams of the ratio $\bar{\rho}/\rho_2$ are given for $y=0$, where $\bar{\rho}$ is the apparent resistivity defined by $(\partial u/\partial x)I_0$. The three dimensional problem where asymptotically the current has also a component parallel to the axis of the cylinder is easily reduced to the previous case. Isopotential lines are plotted for $y=0$. *V. Bargmann* (Princeton, N. J.).

Banerjee, S. S. and Tiwari, S. Y. Shunt excited broadcasting antenna. Indian J. Phys. **16**, 337–342 (1942). [MF 8235]

The shunt excited antenna is essentially a vertical antenna, efficiently grounded and excited at a point on it at a suitable distance above the ground. The present paper studies the intensity of the field radiated from such an antenna, with various heights of the excitation point. The optimum heights for several special cases are indicated. *R. M. Foster* (New York, N. Y.).

Schelkunoff, S. A. The impedance of a transverse wire in a rectangular wave guide. Quart. Appl. Math. **1**, 78–85 (1943). [MF 8190]

Approximate formulae are derived for the impedance of a transverse wire extending from side to side of a rectangular wave guide, the radius of the wire being small, the current uniformly distributed over the wire and the wave guide infinitely long on both sides of the wire. The internal impedance of the wire is given by the usual theory. The external impedance is obtained as an infinite series, each term of the series being associated with an individual TE (transverse electric) wave generated by the current in the wire. The series thus obtained converges slowly; it is shown how to transform this series into a more rapidly convergent series. The analysis is extended to include the case of a split wire, that is, a transverse wire extending from side to side of a rectangular wave guide as before, but broken by a short gap in the wire. *R. M. Foster* (New York, N. Y.).

March, Arthur. Raum, Zeit und Naturgesetze. Naturwissenschaften **31**, 49–59 (1943). [MF 8239]

In this article the author investigates the connection between geometry and microphysics. [The clarification of this problem was obtained in collaboration with E. Foradori [Z. Phys. **114**, 653–666 (1939); **115**, 245–256 (1940); these Rev. **1**, 184, 352].] To apply geometry to reality, one has to assume that matter consists of point particles and that coincidences of point particles occur. But what we observe are coincidences between real particles. The process of measuring distances by these coincidences is a physical process and its analysis must be based upon physical and not geometrical concepts. The author's fundamental assumption is that it is impossible, by any observations, to associate different space coordinates with two resting particles when their distance apart is smaller than l_0. [Unfortunately the author does not explain the meaning of the expression "distance smaller than l_0."] By a similar assumption $t_0=l_0/c$ is considered to be the smallest measurable time interval. The last part of the article analyzes the connection between these assumptions and experiment. By using the uncertainty principle, the inequality

$$||\Delta p|^2-(\Delta E/c)^2|\leqq h/2l_0$$

is deduced [the notation is obvious]. This inequality, which now seems to be generally accepted, is consistent with what we know about the creation of showers. The next conclusion is that this theory eliminates the infinities which appear in quantum electrodynamics. The particles have finite energies because their dimensions must be finite and the energy of radiation is finite because the spectrum is cut out for $\lambda<4l_0$. The length l_0 also plays an important role

in the nuclear theory. By a simple calculation based on less simple assumptions of nuclear theory, the author deduces the numerical value for l_0 and finds it to be of the order of the radius of the electron. *L. Infeld* (Toronto, Ont.).

Heitler, W. On the particle equation of the meson. Proc. Roy. Irish Acad. Sect. A. **49**, 1–28 (1943). [MF 8359]

Kemmer's field equations of the meson theory have the disadvantage that the wave function has redundant components [Kemmer, Proc. Roy. Soc. London. Ser. A. **173**, 91–116 (1939); these Rev. **1**, 95]. Sakata and Taketani derived a new wave equation from Kemmer's by eliminating these redundant components [Sci. Papers Inst. Phys. Chem. Res. **38**, 1 (1940); Proc. Phys.-Math. Soc. Japan (3) **22**, 757 (1940)]. The present author rederives and discusses the results of Sakata and Taketani in fuller detail.
A. Schwartz (State College, Pa.).

Jauch, J. M. Meson theory of the magnetic moment of proton and neutron. Phys. Rev. (2) **63**, 334–342 (1943). [MF 8338]

The magnetic moment of the proton and neutron is calculated to the second order in the perturbation method. The momentum integrals which diverge in the usual theory are made convergent by using the λ-limiting process of Wentzel and the negative energy hypothesis of Dirac. Both the pseudoscalar meson and the usual vector meson with unit gyromagnetic ratio give the same result: the correct order of magnitude but the wrong sign. Agreement with observed values is shown to be possible by assuming an anomalous magnetic moment of about 3 meson magnetons for the vector meson. *S. Kusaka* (Princeton, N. J.).

Ives, Herbert E. Impact of a wave-packet and a reflecting particle. J. Opt. Soc. Amer. **33**, 163–166 (1943). [MF 8117]

The collision of a wave-packet of definite length with a reflecting particle is considered using classical wave-theory only. The formulae for the change of frequency and of energy-density for the wave-packet on collision with a mirror moving with constant velocity are assumed. It is further assumed that the energy-density for a collision with a mirror initially at rest and which is set in motion by the collision will be the same as the formula for the uniformly moving mirror except that the average velocity of the mirror will replace the uniform velocity. The loss of energy between incoming and reflected wave-packet is equated to the work done by the mirror moving against the pressure, though it is not explained by what this pressure is exerted. The impact equation for the wave-packet during the collision gives the mass-equivalent of the packet (of amount $1/c^2$) and the equations for the conservation of mass and momentum for the collision of wave-packet and reflecting particle provide a formula for the variation of mass with velocity of the particle. This formula is said, on the strength of a series of successive approximations, to approach asymptotically the Lorentz form $M(1-v^2/c^2)^{-\frac{1}{2}}$, where M is the mass and v the velocity of the particle.
G. C. McVittie (London).

Smorodinskij, J. The "Bremsstrahlung" of the particle with spin one. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **12**, 181–198 (1942). (Russian) [MF 7590]

The "Bremsstrahlung" of a mesotron of spin one (electromagnetic radiation in the Coulomb field of a nucleus with charge Ze) has been previously treated by F. Boothe and A. H. Wilson [Proc. Roy. Soc. London. Ser. A. **175**, 483 (1940)] and by M. Kobayasi and R. Utiyama [Sci. Papers Inst. Phys. Chem. Research **37**, 221 (1940); Proc. Phys.-Math. Soc. Japan (3) **22**, 882 (1940)], by the method of impact parameters, and by R. F. Christy and S. Kusaka [Phys. Rev. (2) **59**, 405 (1941)], who used the methods of radiation theory and applied Kemmer's form of the mesotron equations. It is the aim of Smorodinskij's paper to derive the cross section for Bremsstrahlung (and for the creation of mesotron pairs by photons) and, in particular, to discuss in greater detail the case of high mesotron energies. Christy and Kusaka [in the paper quoted above] have already pointed out that the method of impact parameters, which is fairly accurate in the case of the electron, cannot be equally well applied to the mesotron, the reason being that for the electron impact parameters smaller than the Compton wave length may be neglected whereas in the case of a particle with spin one the main contribution is due to the smallest parameters, for which, however, this method is rather unreliable.

Starting from Proca's equation for the mesotron in an external electromagnetic field, the author computes by a first order perturbation method the mesotron wave function and the four vector current density, which gives rise to an electromagnetic radiation. The potentials A_μ describing the electromagnetic field of the nucleus (which is treated as a point charge) are decomposed into their Fourier components; the intensity of the radiation and the probability of emitting a photon are first computed for every Fourier component ("pseudophoton") and the total cross section is then obtained by integrating over all pseudophotons. It is assumed that the energy of the mesotron is large compared to its rest energy. Finally the following modifications are introduced. (a) In a paper dealing with the Compton effect of the mesotron, L. Landau and J. Smorodinskij [Acad. Sci. USSR. J. Phys. **4**, 455 (1941)] have shown that for the computation of the cross section one has to omit those processes for which (in the system where the mesotron is initially at rest) the energy of the scattered photon is greater than $\mu c^2/\alpha$ (μ mesotron mass, $\alpha = e^2/\hbar c$). For the interaction of the mesotron with pseudophotons the author uses the same criterion in the relativistically invariant form $-k_i^0 k_i^1 \gg (\mu c)^2/\alpha$, where k_i^0 and k_i^1 are the energy-momentum vectors of mesotron and emitted photon, respectively, and hence cuts out pseudophotons with very high energies. (b) Finite size of the nucleus: the charge of the nucleus is assumed to be distributed over a sphere of radius R, and processes occurring within this sphere give rise to a cross section proportional to Z (instead of Z^2). (c) For high energies the recoil of the nucleus should be taken into account; it is shown, however, that this would change the computed cross section only by a factor of the order of magnitude 1.

Results. Denote the mesotron mass by μ, its energy by E and the nuclear radius by R. The total cross section for Bremsstrahlung is then given by

$$\sigma = Zr_0^2\left[\frac{1}{6}\left(\frac{E}{Mc^2}\right)^2 + aZ\frac{E}{Mc^2}\cdot\frac{\hbar}{McR}\right], \qquad \mu c^2 \ll E \ll Mc^2,$$

$$\sigma = Zr_0^2\left[b\ln^2\frac{E}{Mc^2} + aZ\cdot\frac{E}{Mc^2}\cdot\frac{\hbar}{McR}\right], \qquad Mc^2 \ll E \ll Mc^2\frac{McR}{\hbar},$$

$$\sigma = Zr_0^2\cdot b\left[\ln^2\frac{E}{Mc^2} + Z\cdot\ln^2\frac{E}{Mc^2}\cdot\frac{\hbar}{McR}\right], \qquad Mc^2\cdot\frac{McR}{\hbar} \ll E,$$

where $M = \mu/\alpha^{\frac{1}{2}}$, $r_0 = e^2/\mu c^2$ and a, b are constants of the

order 1. (For a uniform distribution of the nuclear charge $a=2\pi/27$.) For values of Z of the order 10, the leading terms are those proportional to Z^2, which coincide with the corresponding terms in the formulas of Christy and Kusaka (for $E \ll Mc^2 \cdot McR/\hbar$).

In the last section of the paper the author briefly discusses the creation of mesotron pairs by photons, which might be treated in the same way, and refers to the paper of Christy and Kusaka for further details. *V. Bargmann.*

Smorodinsky, J. "Bremsstrahlung" of the particles with a unit spin (mesotrons?). Acad. Sci. USSR. J. Phys. **6**, 264–277 (1942). [MF 8349]

This is the English translation of the paper reviewed above. *V. Bargmann* (Princeton, N. J.).

Langevin, Paul. Sur les chocs entre neutrons rapides et noyaux de masse quelconque. C. R. Acad. Sci. Paris **214**, 517–522 (1942). [MF 7899]

A solution is given for the following problem. A neutron suffers elastic collision with atomic nuclei of some definite mass. Calculate the probability that the neutron falls into an energy interval E to $E+dE$ after an arbitrary number of collisions. The solution is obtained by the evaluation of a volume element in a multi-dimensional phase space. For the case of collision with protons, it reduces to the well-known law that the probability is proportional to dE/E.

L. W. Nordheim (Durham, N. C.).

Bhabha, H. J. and Chakrabarty, S. K. Calculations on the cascade theory with collision loss. Proc. Indian Acad. Sci., Sect. A. **15**, 464–476 (1942). [MF 8179]

A new solution is given for the cascade theory of showers under assumption of an energy-independent energy loss by collision and of the Bethe-Heitler complete screening cross section for radiative processes. The boundary conditions are rigorously fulfilled in contrast to previous treatments of the problem. The solution proceeds in terms of the form

$$P_n(E,t)=(2\pi i E_0)^{-1}(\beta/E_0)^2\int_{\sigma-i\infty}^{\sigma+i\infty}\{E_0/(E+\beta g(s,t))\}^s f_n(s,t)ds,$$

where E_0 is the initial energy, E the energy, β the collision loss, t the depth, all in units of radiation theory. Generally, the first approximation $P_0(E, t)$ for the energy distribution is sufficient. A table for the total number of particles as a function of depth and primary energy is given.

L. W. Nordheim (Durham, N. C.).

Chakrabarty, S. K. Accurate calculations on the cascade theory of electronic showers without collision loss. Proc. Nat. Inst. Sci. India **8**, 331–337 (1942). [MF 8178]

The formulae of the companion paper by Bhabha and Chakrabarty [see the preceding review] are evaluated neglecting collision losses for a primary electron and a primary photon. The two distributions are very similar, except that photon produced showers are shifted to greater depths about one unit length. The weaknesses of previous treatments are pointed out and are illustrated by a comparison of the numerical results of the new method with those of other authors. *L. W. Nordheim.*

Proca, Alexandre. Sur la théorie des particules matérielles et en particulier sur les électrons de spin $\frac{1}{2}$. C. R. Acad. Sci. Paris **214**, 606–607 (1942). [MF 7895]

It is noted that the relativistic invariant wave equation for a particle with spin $\frac{1}{2}$ is given by the following generalization of Dirac's equation

$$\gamma^\mu\partial\psi/\partial x^\mu+(mc/\hbar)\psi-\lambda(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)x_\mu\partial\psi/\partial x^\nu=0,$$

where the γ^μ are the usual relativistic Dirac operators and $1/\lambda$ a new universal length. *L. W. Nordheim.*

Ginsburg, V. L. The relativistic theory of excited spin states of the proton and the neutron. Phys. Rev. (2) **63**, 1–12 (1943). [MF 7818]

A wave equation is derived which describes a particle which, in the absence of external actions, can be either in a state of spin $\frac{1}{2}$ or in a state of spin $\frac{3}{2}$ and a different rest mass. The interaction of such particles with the electromagnetic field can be treated according to normal methods; this is also the case when the particles possess an extra magnetic moment. The scattering of light is treated for such particles and it is shown that the presence of the higher spin state prevents an unlimited increase of the scattering cross section at high energies, which would occur otherwise. *L. W. Nordheim* (Durham, N. C.).

Lifshiz, I. M. Optical behaviour of non-ideal crystalline lattices in the infra-red region. I, II, III. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **12**, 117–180 (1942). (Russian) [MF 7865]

Two types of small deviations from lattice periodicity are considered: (1) arbitrary concentrations of isotopes and (2) small concentrations of arbitrary impurities. The problem treated consists in the determination of the electric moment $\mathbf{P}$ of the crystal excited by the field $\mathbf{E}\exp(i\omega t)$ and is reduced substantially to the determination of a vector $\mathbf{u}$ satisfying the equation

$$(*)\qquad (A-\omega^2+\Omega)\mathbf{u}=\boldsymbol{\phi}+\boldsymbol{\delta},$$

where A is a linear operator which is given as a function of the coefficients of the expansion of the total energy in power series of the atom coordinates (the expansion being limited to the terms of second order), $\mathbf{u}$ is a function of atom masses and their amplitudes, $\boldsymbol{\phi}$ is a function of $\mathbf{E}$ and of the electric charges of the atoms. The operator Ω and the vector $\boldsymbol{\delta}$ are expressions of the imperfect periodicity of the lattice. The length of the excitation wave is assumed sufficiently large, so that the phase difference between the various atoms is neglected. In the case in which the periodicity perturbation is due to the presence of isotopes, $\boldsymbol{\delta}$ is zero and Ω is a diagonal matrix, whose elements are determined in terms of ω, of the masses of isotopes and their relative concentrations. For simplicity the author considers here the case of two isotopes only. An explicit expression for $\mathbf{u}$ is given in the ideal case ($\Omega=0$, $\boldsymbol{\delta}=0$) by using an integral representation of an arbitrary analytic function of A in terms of the roots of a secular equation related to A. In the case of isotopes, because of the small magnitude of the elements of Ω, $\mathbf{u}$ is put in the form

$$(\dagger)\qquad \sum_n(-1)^n[(A-\omega^2)^{-1}\Omega]^n\mathbf{u}_0,$$

$\mathbf{u}_0$ being the expression of $\mathbf{u}$ in the ideal case. From this a general expression of $\mathbf{P}$ is obtained. In the case of small concentration of impurities of any kind, the mean value of $\mathbf{P}$ is expanded in power series of the concentrations and the equation (*) is split into two equations $(A-\omega^2+\Omega)\mathbf{u}_1=\boldsymbol{\phi}$, $(A-\omega^2+\Omega)\mathbf{u}_2=\boldsymbol{\delta}$, with $\mathbf{u}=\mathbf{u}_1+\mathbf{u}_2$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are put into a form similar to (†). The paper contains a detailed discussion of the range of validity of the expansions and explicit formulas in particular cases. *I. Opatowski.*

and

pin
63,

icle
n a
rest
tro-
ds;
tra
for
the
the
cur
).

line
auk
12).

city
and
The
tric
and
r u

n of
in
eing
of
and
the
the
suf-
the
the
pes,
are
heir
lers
sion
an
n of
A.
e of

is a
nall
e of
the
$= \phi$,
put
iled
and